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Neologicism, Frege's Constraint, and the Frege-Heck Condition

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Abstract

One of the more distinctive features of Bob Hale and Crispin Wright's neologicism about arithmetic is their invocation of Frege's Constraint – roughly, the requirement that the core empirical applications for a class of numbers be "built directly into" their formal characterization. In particular, they maintain that, if adopted, Frege's Constraint adjudicates in favor of their preferred foundation – Hume's Principle – and against alternatives, such as the Dedekind-Peano axioms. In what follows we establish two main claims. First, we show that, if sound, Hale and Wright's arguments for Frege's Constraint at most establish a version on which the relevant application of the naturals is transitive counting – roughly, the counting procedure by which numerals are used to answer "how many"-questions. Second, we show that this version of Frege's Constraint fails to adjudicate in favor of Hume's Principle. If this is the version of Frege's Constraint that a foundation for arithmetic must respect, then Hume's Principle no more – and no less – meets the requirement than the Dedekind-Peano axioms do.

Introduction

There are different formal characterizations of the natural numbers available: the Dedekind-Peano Axioms, Hume's Principle, and Øystein Linnebo's [2009] abstraction principle ("2L-$\mathbb{N}_0$"), to name a few. One *egalitarian* attitude towards this proliferation is that they are simply different, equally legitimate ways of codifying the *same things*, the natural numbers. Many forms of structuralism in the philosophy of mathematics exhibit such an attitude. Roughly put, so long as a formal characterization specifies the relevant structure – an ω-sequence – it succeeds in this task. And since there are many ways of specifying an ω-sequence, there are many equally good ways of characterizing the natural numbers. None is in any non-prudential sense better than another. Moreover, none is in any interesting sense more *fundamental* than any other.

There is another, *non-egalitarian*, attitude towards such matters. Even if there are many correct formal characterizations of the naturals, one is more *fundamental* than the others in that it provides the *correct* foundations for (basic) arithmetic. Though there are different versions of this approach, the abstraction-based neologicism of Crispin Wright and Bob Hale is surely the most prominent and influential.

Central to Hale and Wright's neologicism is the claim that Hume's Principle (HP) – not the Dedekind-Peano (DP) axioms, or any other formal characterization – constitutes the *correct* foundations for arithmetic. A centerpiece of their argument for this claim is a requirement known as *Frege's* (*Application*) *Constraint* – roughly, the thesis that a satisfactory foundation for a mathematical theory – in the present case, basic arithmetic – must somehow "build in" its primary empirical applications. Hale and Wright have sought to justify the adoption of Frege's Constraint for basic arithmetic, and to argue that this constraint, appropriately formulated, adjudicates in favor of assigning a foundational status to HP.

We argue that Hale and Wright's approach fails. In particular, their arguments fail to justify a version of Frege's Constraint that adjudicates in favor of HP, since either the DP Axioms equally well meet the version of Frege's Constraint for which they argue, or else neither characterization does.

The paper is organized as follows. §1 outlines some of the core background to the present issue. We sketch the theoretical context in which Hale and Wright develop their neologicism, and remind the reader of the main formal machinery on which they rely. We also explain how their view is beholden to capturing accurately our ordinary concepts of natural numbers, via what we call "the Frege-Heck Condition", and how this condition underlies the adoption of Frege's Constraint. §2 sketches what we take to be the two most plausible extant arguments for Frege's Constraint, due to Wright [2000] and Hale [2016]. Although we offer some critical remarks, our main aim is to highlight the fact that both arguments, if sound, would deliver a version of Frege's Constraint on which *transitive counting* is the core empirical application relevant to satisfying Frege's Constraint. Finally, in §3, we show that if *this* is the version of Frege's Constraint that we should endorse, then it fails to adjudicate in favor of adopting HP as fundamental. When it comes to meeting Frege's Constraint, HP fairs no better or worse than the DP axioms.

1. Current Philosophy of Mathematics and Neologicism

1.1 The Philosophy of What Mathematics?

Much contemporary work in the philosophy of mathematics focuses on what is practiced by professional mathematicians, or at least the mathematics taught in graduate and advanced undergraduate courses. Philosophers address ontological, epistemological, and methodological issues concerning branches like number theory, real analysis, complex analysis, functional analysis, and the like. Much, but not all, of this work concerns so-called foundational branches of mathematics, such as set theory, category theory, and type theory.

Contemporary philosophers are also keenly interested in the *applications* of mathematics. Typically, this concerns the use of mathematics in science, such as its role in expressing scientific laws and the place of differential equations in physics. There is less concern, overall, with the role and place of elementary mathematics in everyday life, such as the use of numbers to count collections, to balance checkbooks, and to measure lengths and volumes. Of course, it is assumed that there is *some* connection between the everyday use of numbers to count and measure and the use of numbers in advanced mathematics, but the exact nature of this relationship is not often queried.

1.2 Abstractionist Neologicism

There are, however, some notable exceptions to this trend. Perhaps the most prominent is the program of *abstractionist neologicism*, which began with Wright [1983] and was extended with Hale [1987]. It continues through many extensions, objections, and replies to objections (see Hale and Wright [2001]).

The overall plan is to develop branches of established mathematics using abstraction principles in the form

$$\text{(ABS)} \quad \forall a \forall b\, (\Sigma\,(a) = \Sigma\,(b) \equiv E\,(a, b))\,,$$

where a and b are variables of a given type (typically first-order, ranging over individual objects, or second-order, ranging over concepts or properties), Σ is a higher-order operator, denoting a function from items of the given type to objects in the range of the variables, and E is an equivalence relation over items of the given type. Frege ([1884], [1893]) employed three such abstraction principles. One of them, used for illustration, comes from geometry:

> The direction of l_1 is identical to the direction of l_2 if and only if l_1 is parallel to l_2.

A second is the infamous Basic Law V:

$$(\forall F)(\forall G)[\mathbf{Ext}F \;=\; \mathbf{Ext}G \;\equiv\; \forall x(Fx \equiv Gx)],$$

which, of course, is inconsistent in standard logical systems. A third was dubbed $N^=$ in Wright [1983] and is now called *Hume's Principle*:

$$\text{(HP)} \quad (\forall F)(\forall G)[\#F \;= \#G \equiv \; F \approx G],$$

where # is the number of-operator, and $F{\approx}G$ is an abbreviation of the second-order statement that there is a one-to-one relation mapping the F's onto the G's. In other words, HP states that for any concepts F and G, the number of Fs is identical to the number of Gs if and only if the Fs are equinumerous with the Gs. Georg Cantor deployed this principle, albeit not formulated as rigorously, to obtain extensive and profound results concerning infinite cardinals. Our concern here is only with the familiar natural numbers.

As is now well-known, Frege's *Grundlagen* [1884] and *Grundgesetze* [1893] contain the essentials of a derivation of the DP axioms from HP. Indeed, in [1893], the only essential use of the inconsistent Basic Law V is to derive the two conditionals in Hume's Principle. The rest of the derivation follows from those two conditionals.[1]

This derivation, now called *Frege's Theorem*, reveals that Hume's Principle entails that there are infinitely many natural numbers. The development of arithmetic from HP is sometimes called *Frege Arithmetic*. This theory is taken to be the first success story of the abstractionist program.

Some of the details matter here. Frege defines *zero* to be the number of the concept *being non-self-identical*:

$$0 = \#[\lambda x(x \neq x)].$$

It follows that zero is the number of any concept that does not apply to anything. Next comes Frege's [1884, §76] definition of the successor relation among cardinal numbers. In modern notation, n is the *cardinal-successor* of m just in case:

$$\exists F \exists G(m = \#F \& n = \#G \& \exists x(Gx \& \forall y(Fy \equiv (y \neq x \& Gy))))$$

The same definition is employed in Frege's *Grundgesetze* §43 and in Wright's [1983] book that launched the abstractionist program. Clearly, the idea is that n is a successor of m just in case: if we have a collection with m members and add a new member, it will have n members.

Of course, the number one is the successor of zero, two is the successor of one, etc. A *natural number* is then defined to be an ancestor of zero under the successor relation (using Frege's definition of the ancestral of a relation). The DP Axioms then follow: zero is a natural number; each natural number has a unique successor; the successor relation is one-to-one; zero is not the successor of any natural number; every natural number except zero is the successor of a unique natural number; and an induction principle holds among the natural numbers: for any property P, if P holds of zero, and if for any natural number n, if P holds of n, and if P holds of the successor of n, then P holds of all natural numbers.

Although the search is on to develop more advanced branches of mathematics on the basis of abstraction principles, our present concern is only with the arithmetic of the natural numbers. One of the explicit goals of Frege's logicism is to establish that arithmetic and real analysis are analytic and, thus, not founded on Kantian intuition. Wright and Hale adopt a similar goal. Wright [1997, 210] concedes that HP may not amount to an explicit definition of "cardinal number" or of "identity of cardinal number", in some strict sense of definition. Nevertheless,

> Frege's theorem will still ensure... that the fundamental laws of arithmetic can be derived within a system of second-order logic augmented by a principle whose role is to *explain*, if not exactly to define, the general notion of identity of cardinal number, and that this explanation proceeds in terms of a notion which can be defined in terms of second-order logic. If such an explanatory principle... can be regarded as *analytic*, then that should suffice... to demonstrate the analyticity of arithmetic.

The rough idea is that HP implicitly defines the notion of a *cardinal number* by providing identity conditions for using the number of-operator (#).[2] The DP axioms are then logical consequences of HP, given the relevant definitions, and are thus themselves analytic.

1.3 The Frege-Heck Condition

So far we have sketched the manner in which the abstractionist program seeks to develop established branches of mathematics from abstraction principles, such as HP. So described, the conditions on successfully executing the program for arithmetic may appear merely formal: to derive from abstraction principles something that has the form of the DP axioms.

As Richard Heck [1997] points out, however, this would be a misunderstanding of the program. In addition to this formal achievement, one must also show that HP actually characterizes *the natural numbers*. It is not enough merely to specify some abstraction principle or other as basic and then derive something that has the form of the DP axioms. In addition, we need some assurance that this suffices for the derivation of the *very same* DP axioms that we all know and some of us love. In particular, one needs some justification that the results of the abstraction principle and the definitions have the appropriate *contents*. That is, the derivations must be *about* the natural numbers, not something merely isomorphic to them. As Heck puts it:

> What is required if logicism is to be vindicated is not just that there is *some conceptual truth or other* from which what *look like* axioms for arithmetic follow, given certain definitions: That would not show that the truths of arithmetic, *as we ordinarily understand them*, are analytic, but only that arithmetic can be interpreted in some analytically true theory. To put the point differently, if we are so much as to evaluate logicism, we must first uncover the 'basic laws of arithmetic', laws which are not just sufficient to allow us to prove translations of arithmetical truths, but laws from which arithmetical truths themselves can be proven. (The distinction is not a mathematical one, but a philosophical one.) (Heck [1997, 596–97])

Heck provides an illuminating illustration of the point. Frege surely knew that Euclidean geometry can be interpreted within real analysis: one defines a "point" to be a pair of real numbers, and then establishes translations of the Euclidean axioms. Frege held at the time that real analysis, like arithmetic, is analytic, but he did not hold that geometry is analytic, adopting instead the more traditional Kantian view that this branch of mathematics is synthetic *a priori*. So, one can conclude, for the Fregean to establish that a given theory is analytic (or all but analytic), it does not suffice merely to interpret the theory within an analytically true theory.

The abstractionists accept this commitment (see, for example, Wright [2000, 323]). They adopt the burden of showing that by starting with HP and then adopting the above Fregean definitions of zero, successor, and natural number, one has met what we call the *Frege-Heck Condition*:

> A satisfactory logicist derivation of arithmetic should consist in the derivation of truths with the appropriate contents – viz. they should be *about the natural numbers* as ordinarily understood, and not merely some isomorphic surrogate.

But how does one discharge this burden? Here is where the abstractionist makes contact with the everyday notion of number – the one used to count toes and balance checkbooks. The claim is that by deriving Frege's Theorem, one has delivered the very same natural numbers as those used in everyday life. But what are those? Here is where issues of applicability enter the picture.

1.4 Frege on Applicability

Frege was keenly interested in how mathematical theories are applied, and objected to other foundational theories on the grounds that they failed to explain empirical applications in the right kind of way. For our purposes, Frege's discussion of the real numbers is especially prescient. Frege took himself to be forging a synthesis between the old, geometric approaches and the modern, axiomatic approaches traced to Weierstrass, Dedekind, and Cantor. Discussing his own view, he writes:

> ... we avoid the emerging problems of the [modern] approaches, that either measurement does not feature at all, or that it features without any internal connection grounded in the nature of the number itself, but is merely tacked on externally, from which it follows that we would, strictly speaking, have to state specifically for each kind of magnitude how it should be measured, and how a number is thereby obtained. Any general criteria for where the numbers can be used as measuring numbers and what shape their application will then take, are here entirely lacking ...
>
> One may surely expect arithmetic to present the ways in which arithmetic is applied, even though the application itself is not its subject matter. (Frege [1903, §159])

Here, "arithmetic" is the theory of the finite cardinal numbers and the theory of the real numbers. Frege insisted that a proper foundation for real analysis build the primary applications of the real numbers into the very fabric of the theory. Applications should not be merely "tacked on externally", as they are in the celebrated accounts of Dedekind and Cantor (via what is now known as measurement theory). Presumably, the same goes for elementary arithmetic.

1.5 Frege's Constraint

The requirement that the primary empirical applications of a class of numbers be "built directly into" their characterization has come to be known as *Frege's (Application) Constraint*, and its endorsement is amongst the more distinctive features of Hale-Wright abstractionism. They argue that their view satisfies the Frege-Heck condition because it meets Frege's Constraint for elementary arithmetic. Moreover, they argue, their chief rivals do not meet Frege's Constraint, and for that reason, deliver, at best, a structure isomorphic to, but distinct from, the naturals.

Crispin Wright [2000, 324] articulates a broad version of the Constraint well:

> ... a satisfactory foundation for a mathematical theory must somehow build its applications, actual and potential, into its core – into the content it ascribes

> to the statements of the theory – rather than merely 'patch them on from the outside'.

With a little more detail:

> What is it to observe Frege's constraint? To insist that the general principle governing the application of a type of number be built into their characterization from the start is in effect just to insist such numbers be characterized by reference to a principle which explains what kind of entities they apply to – are of – and what it is for such entities to be associated with the same or different such numbers . . . To view such principles as philosophically and mathematically foundational is accordingly to view the applications of the sorts of mathematical objects they concern as belonging to the essence of objects of those sorts. (p. 325)

Michael Dummett [1991] also articulates and, it seems, endorses Frege's Constraint, with respect to both the natural numbers and the real numbers. This applies even to the most elementary, or basic, applications.

> Any specific type of application will involve empirical, or at least non-logical, concepts alien to arithmetic; . . . To make such applications intrinsic to the sense of arithmetical propositions is therefore to import into their content something foreign to it, . . . What is intrinsic to their sense, however, is the general principle governing all possible applications. That must accordingly be incorporated into the definitions of the fundamental arithmetical notions. It is not enough that they be defined in such a way that the possibility of these applications is subsequently provable; since their capacity to be applied in these ways is of their essence, the definitions must be so framed as to display that capacity explicitly. (Dummett [1991, 60])

Emphasis here is on *subsequent provability*: because the empirical applications of arithmetic are *essential* to the naturals, those applications must be directly reflected in the principle(s) characterizing those numbers, not subsequently derived from them.

With a bit of hindsight, perhaps, both Dedekind's [1872] account of the real numbers and his later account of the natural numbers [1888] are broadly structuralist (see Shapiro [1997, Chapter 5]). Dedekind shows how the natural numbers relate to each other, and he shows how the real numbers relate to each other, by giving a categorical characterization of each structure, the DP axioms in the case of the natural numbers. Further, Dedekind has no trouble showing how both kinds of numbers are typically applied. Specifically, as we will see in §3, Dedekind is able to show how to relate the cardinality of various collections to initial segments of the natural numbers.

However, for Frege – and, it seems, for Dummett, Wright, and Hale – this explanation of how the naturals can be applied to collections comes too late. For the explanation is "tacked on externally". To quote Dummett again:

> The identity of a mathematical object may sometimes be fixed by its relation to what lies outside the structure to which it belongs; what is constitutive of

> the number 3 is not its position in any progression whatsoever, or even in some particular progression, nor yet the result of adding 3 to another number, or of multiplying by 3, but something more fundamental than any of these: the fact that, if certain objects are counted 'One, two, three', or, equally, 'Nought, one, two', then there are 3 of them. The point is so simple that it needs a sophisticated intellect to overlook it; and it shows Frege to have been right, as against Dedekind, to have made the use of the natural numbers as finite cardinals intrinsic to their characterization. (Dummett [1991, 53])

Pace Dedekind, then, Dummett insists that the natural numbers should respect Frege's Constraint. Since the empirical application of the natural numbers is intrinsic to them, this fact should be directly reflected in their characterization, not subsequently derived from that characterization.

2. Why Respect Frege's Constraint for Arithmetic?

We have seen that abstractionists suppose that a satisfactory foundation for arithmetic must satisfy the Frege-Heck Condition. Further, we have seen that satisfying Frege's Constraint is supposed to establish that the abstractionists' preferred foundation meets this Condition. Yet this use of Frege's Constraint is, of course, predicated on a pair of assumptions:

1. *Legitimation*: The requirement that a theory of arithmetic should respect Frege's Constraint requires justification.
2. *Adjudication*: It needs to be that the abstractionist program respects Frege's Constraint, whereas the alternatives do not.

The role of Legitimation should be obvious, especially since prominent alternatives to abstractionism – e.g. various versions of structuralism – do not typically endorse Frege's Constraint.[3] The role of Adjudication is similarly obvious. If abstractionism fares no better in meeting Frege's Constraint than the alternatives, then it will not establish HP as the fundamental foundation for elementary arithmetic.[4]

With these requirements in mind, our main complaint can be stated as follows: We know of no good reason to suppose that both Legitimation and Adjudication are satisfied. Rather, the best extant efforts at Legitimation suffer from one or both of the following problems: either they fail to justify the adoption of Frege's Constraint in any form, or else they render Frege's Constraint plausible only at the expense of importing assumptions which render it impossible for abstractionists to satisfy the Constraint.

In what follows, we start the task of developing this complaint by considering what we take to be the best arguments for respecting Frege's Constraint in the domain of elementary arithmetic – the first due to Hale [2016], the second due to Wright [2000]. Our primary goal is to identify what they take to be the primary empirical application of arithmetic, the application that must be "built in".

2.1 Hale's Argument

Hale [2016]'s strategy is to argue that mastery of a certain core application is essential to the possession of natural number concepts and, hence, that a satisfactory definition of natural numbers should somehow build this in directly. This core application is what Benacerraf [1965] calls *transitive counting* – a kind of procedure for answering 'how many'-questions – and stands opposed to what Benacerraf calls *intransitive counting* – reciting the numerals in their canonical order ('one', 'two', etc.). More specifically, transitive counting is the procedure of associating the members of a finite set of objects with an initial sequence of the numerals starting with 'one', thereby indirectly establishing a correspondence between the members of the set and an initial segment of the natural numbers.

> To [transitively] count the members of a set is to determine the cardinality of a set. It is to establish that a certain relation C obtains between the set and one of the numbers – that is, one of the elements of $\mathbb{N}$ . . . Practically speaking, and in simple cases, one determines that a set has k elements by taking (sometimes metaphorically) its elements one by one as we say the numbers one by one (starting with 1 and in order of magnitude, the last number we say being k). To count the members of some k-membered set b is to establish a one-to-one correspondence between the elements of b and the elements of $\mathbb{N}$ less than or equal to k. (Benacerraf [1965, 274])

Intuitively, if there is a one-to-one correlation between the Fs and a sequence of numerals 1, . . . , k, then the numeral k answers the question 'How many Fs are there?'. In other words, it designates the cardinality of the set in question. For example, to transitively count five Elmos is to correlate each member of the set of Elmos with a unique numeral in the series "1, . . . , 5", thus indirectly establishing a one-to-one correspondence between the Elmos and the first five natural numbers.

With this distinction between transitive and intransitive counting in place, Hale's argument proceeds via three claims:

> P1: One who has learned to count both intransitively and transitively, but not yet to add, multiply, etc., has at least a basic grasp of the concepts of the natural numbers.

> P2: It is possible that a trainee – which we'll call the *DP Novice*– should learn to count intransitively, but not transitively, and then proceed directly to learn to do arithmetic. In particular, she might digest the DP axioms, and thus "be introduced to the successor operation . . . and be taught to add, multiply, etc., perhaps by being given the usual recursive definitions of + and x, or perhaps by means of tables".

> P3: The DP-Novice would not yet have a basic grasp of (the concepts of) the natural numbers.

Hale maintains that each of these claims is individually plausible, and that collectively they invite the conclusion that:

> C1: "The possession of the concepts of natural numbers requires understanding their use in transitive counting."

Further, Hale claims that if this is so, then "the fact that the natural numbers can be used to count collections of things is no mere accidental feature, but is *essential* to them" (Hale [2016, p. 340]). Roughly, it is a conceptual necessity that they can be so used. Hence:

> C2: "A satisfactory definition of the natural numbers – a characterisation of what they essentially are – should reflect or incorporate that fact."

In sum, the argument is that a satisfactory characterization of the natural numbers should build their essential empirical application – transitive counting – into that characterization. And this is, of course, precisely what Frege's Constraint requires for arithmetic, assuming the essential application is transitive counting.

Although Hale's argument is a *prima facie* compelling one, we have argued at length elsewhere[5] that it fails Legitimation, for at least two reasons. Very briefly:

a) The inference from C1 to C2 is problematic because it is not generally the case that what's essential to the *possession* of a concept of *X's* is also essential to *X*'s themselves.[6]
b) We find P3 far from obvious, and therefore in need of further defense.[7]

However, we will not dwell on the issue of Legitimation here. For present purposes, the key point is that Hale's argument would, if sound, be an argument for a version of Frege's Constraint that makes transitive counting the core application. As we will see, this means that it delivers a version of Frege's Constraint that fails Adjudication.

2.3 Wright's Argument

We now turn to Wright's [2000] argument for Frege's Constraint. Our primary goal is to show that Wright, too, is committed to transitive counting being at least one core application that must be built directly in to any philosophically satisfactory account of the natural numbers.

Recall Wright's initial gloss of the Constraint:

> To insist that the general principle governing the application of a type of number be built into their characterization from the start is in effect just to insist such numbers be characterized by reference to a principle which explains what kind of entities they apply to – are of – and what it is for such entities to be associated with the same or different such numbers ... (p. 325)

Clearly, this requirement needs Legitimation, if it is to play a substantive role in justifying a foundation for arithmetic. The alternative would be tantamount to the contentious and unsupported insistence that Frege's Constraint must be respected. Still worse, the version of the Constraint that Wright articulates is one that, conveniently, demands precisely what HP delivers: a principle that specifies the sorts of things to which numbers apply, and when such numbers are the same or distinct.

Of course, Wright is fully aware of this, and consequently seeks to *argue for* Frege's Constraint. Although the general contours of his argument are discernable, it is far from obvious – at least to us – how best to render the details. We proceed as follows: In 2.3.1, we outline Wright's general argumentative strategy. Then in 2.3.2 we present a plausible, and more precise reconstruction of his argument.

2.3.1 Wright's Argumentative Strategy

As Wright sees it, the primary resistance to Frege's Constraint comes from structuralists who typically decline to respect it. Thus, Wright's argument for Frege's Constraint is also represented as an argument against structuralism. The argument is broadly transcendental in character. He maintains that *unless* elementary arithmetic satisfied Frege's Constraint, a kind of *a priori* arithmetic knowledge – which he presumes we possess – would be impossible:

> It seems clear that one kind of epistemic access to ... simple truths of arithmetic proceeds precisely *through* their applications. Someone can – and our children surely typically do – first learn the concepts of elementary arithmetic by a grounding in their simple empirical applications and then, on the basis of the understanding thereby acquired, advance to an *a priori* recognition of simple arithmetic truths. (Wright [2000, 327]).

Consider "a child who reasons on her fingers ... that $4 + 3 = 7$". According to Wright, such a process involves a kind of reflection upon the application of arithmetic concepts, the result of which is the possession of basic arithmetic knowledge, e.g. that $4 + 3 = 7$. This knowledge is not merely empirical knowledge of impure mathematical truths.[8] Rather, his contention is that the child can, via reflection on schematic applications, acquire *a priori* knowledge of *pure* arithmetic truths, and so "there is a kind of *a priori* arithmetical knowledge which flows from an antecedent understanding of the way that arithmetical concepts are applied".

According to Wright, despite its reliance on empirical applications, this kind of knowledge is *a priori* because the role that applications play in the child's reflections – when they use their fingers, a diagram, or a collection of blocks, for example – is strongly analogous to that played by paper and pencil constructions in geometry. But, despite its reliance of constructions, since the geometer's knowledge is *a priori*, according to Wright, we should adopt much the same view of the child's arithmetic knowledge, despite its reliance on empirical applications.

Why deny that structuralism – a view which does not respect Frege's Constraint – can accommodate the above sort of *a priori* knowledge? Crudely put, Wright's contention is that structuralists draw an insufficiently tight connection between the *content* of pure arithmetic propositions and their applications. More precisely, if structuralism were correct, then the content of what one knows when grasping simple arithmetic truths would have nothing as such to do with their applications to collections of things, but rather with abstract structures and their relations. In that case, coming to know *a priori* that $4 + 3 = 7$ would require more than "mere reflection" upon the application of arithmetic concepts to schematic applications. Specifically, it would require *additionally* recognizing structural affinities between the collections under consideration and the natural numbers:

> ... the structuralist reconstruction of this knowledge will involve a representation of its content from which an appreciation of potential application will be an additional step, depending upon an awareness of certain structural affinities.

According to Wright, this reconstruction is problematic because it would be tantamount to "changing the subject":

> [W]hatever the details of her epistemological story about the simplest truths of arithmetic, the content of the knowledge thereby explained will not be that of the knowledge we actually have.

For *that* knowledge—the knowledge we actually have—does not "depend upon an *additional* appreciation of structural affinities." Rather it is merely "grounded in reflection upon sample, or schematic applications." As a consequence, to account for the possibility of such knowledge, it must be that:

> ... [S]imple arithmetic knowledge, so acquired, has to have a content in which the potential for application is absolutely *on the surface*, since the knowledge is induced precisely upon sample, or schematic applications.[9]

So if one assumes that an adequate definition of natural number should satisfy the Frege-Heck condition, then such definitions will need to respect Frege's Constraint.

2.3.2 A Reconstruction

While it is not entirely clear how exactly to render the details of Wright's argument, Hale [2016] provides a reconstruction, which we present below in a minimally modified form so as to more faithfully capture Wright's original intentions:

> P1. We can acquire *a priori* knowledge of simple arithmetic truths by mere reflection on the application of the concepts involved in stating those truths.
>
> P2. If we can acquire *a priori* knowledge of simple arithmetic truths by mere reflection, then a connection with application is built into the content of those simple arithmetic truths.
>
> P3. If structuralism about arithmetic is right, no such connection with application is part of the content of arithmetic truths.
>
> Hence [from P2 and P3]:
>
> C1. If structuralism about arithmetic is right, we cannot acquire *a priori* knowledge of simple arithmetic truths by mere reflection.
>
> Hence [from P1 and C1]:
>
> C2. Structuralism about arithmetic is wrong. (Hale [2016], 334])

Wright's complaint against structuralism is that by additionally requiring the recognition of structural affinities, structuralism fails to account for the *immediacy* of such *a priori* arithmetic knowledge. This is what we mean by *mere reflection* in P1: by stipulation, one who gains arithmetic knowledge by mere reflection does so without having to engage in the further step of recognizing structural affinities between the objects figuring in schematic applications and the natural numbers themselves. According to Wright, no such step is required since the contents of simple arithmetic truths are such that the "potential for application is absolutely *on the surface*". Thus, Wright concludes, the content that structuralists attribute to simple arithmetic statements is not the content of our *actual* arithmetic knowledge. Hence, structuralism violates the Frege-Heck Condition precisely because it fails to respect Frege's Constraint.

2.3.3 Transitive Counting Again

Although Wright's argument is an intriguing one, we find it less than convincing. Specifically, we see no reason to suppose that P1 is true. Certainly, Wright has nothing compelling to say on the matter. Sometimes he appears to treat the claim as if it were just obvious. Thus he says:

> It seems clear that one kind of access to . . . simple truths of arithmetic precisely proceeds *through* their applications.

But if the relevant sort of access is of the merely reflective variety – and so requires no recognition of structural affinities – then it does not seem at all clear that humans have such access to arithmetic truths. Not to us, at any rate.

In other places, Wright appears to think that it is possible to acquire *a priori* arithmetic knowledge via mere reflection on applications because children *actually* acquire such knowledge in this way. Recall Wright's comment:

> Someone can—and *our children surely typically do*—first learn the concepts of elementary arithmetic . . . and then, on the basis of the understanding thereby acquired, advance to an *a priori* recognition of simple arithmetical truths.

Yet it is far from obvious that children do this – especially not if the process is supposed to involve no recognition of structural affinities. Certainly, the extensive developmental literature on mathematical cognition fails to suggest anything of the sort. In fact, some highly influential accounts of how number cognition develops make the recognition of structural affinities a crucial step in the process.[10]

In any case, we propose not to dwell on the soundness of Wright's argument here. For present purposes, the main point we wish to highlight is that, as with Hale's argument, Wright's argument would, if sound, support a version of Frege's Constraint that makes *transitive counting* the core application. First, this is how Hale interprets Wright (Hale, 2016, p. 333). Secondly, recall that the sort of *a priori* arithmetic knowledge acquired via mere reflection by "a child who reasons on her fingers" is supposed to be similar to the sort of geometric knowledge acquired via the construction of diagrams. But clearly "reasoning on one's fingers" *just is* a form of transitive counting. Third, Wright himself suggests as much when discussing the shortcomings of structuralism:

> This suggests a distinction which, wherever it can be upheld, will mandate something close to Frege's Constraint. It is one thing to explain how (*a priori*) knowledge could be acquired of a system which, taken in conjunction with

> certain supplementary reflections, can then be applied in the same ways as an entrenched mathematical theory. But that will not suffice to provide a correct (if idealized) reconstruction of the content of *what we actually know* in knowing that theory if at least some of that knowledge can be achieved just by the reflective exercise of concepts acquired and applied in the ordinary course of counting and calculation… (Wright [2000, 328], emphasis in original)

Here, Wright explicitly connects the content of our actual arithmetic knowledge obtained via mere reflection on the concepts acquired and exercised through *counting*. Again, since this takes the form of "reasoning on one's fingers", at least some of our actual arithmetic knowledge comes via mere reflection on the concepts acquired and exercised through *transitive* counting.

3. Frege's Constraint and the Parity Problem

Let's summarize the discussion so far. In the domain of arithmetic, Frege's Constraint is intended by neologicists to Adjudicate: to show that their preferred foundation – HP – meets the Frege-Heck Condition, whereas the main potential alternatives, especially the DP axioms, do not. For Frege's Constraint to play this role, it requires Legitimation: one must justify the assumption that a theory of arithmetic should respect Frege's Constraint. Finally, the two best arguments for respecting Frege's Constraint for arithmetic, if sound, would legitimate a version of Frege's Constraint on which the core empirical application is transitive counting. Call this version of the Constraint, FC_{tc}.

This sets the stage for our primary objection to neologicist efforts to meet the Frege-Heck Condition by invoking Frege's Constraint:

> *The Parity Problem:* If transitive counting is the relevant application, then Frege's Constraint fails Adjudication. Specifically, it will not be the case that HP respects FC_{tc} whereas the DP Axioms do not. Nor, for that matter, will it be the case that the DP axioms respect FC_{tc} where HP does not. When it comes to transitive counting, the two are on a *par*.

Suppose that transitive counting is the primary empirical application relevant to satisfying Frege's Constraint. Then the principles characterizing the natural numbers ought to "directly reflect or incorporate" – or "build in" – that they can be used this way. But what exactly would it mean to *build in* transitive counting *directly*? The requirement cannot be that the transitive counting *procedure*, itself, be built in since such a procedure is not truth apt or propositional, as axioms generally are. Moreover, it is quite clear that neither HP nor the DP axioms specify a procedure of this sort. In which case, on the present construal, FC_{tc} would obviously fail to Adjudicate.

A more plausible construal of FC_{tc}, and the one we explore here, is that a formal characterization "builds in" transitive counting if it specifies the cognitive or epistemic prerequisites relevant to performing the transitive counting procedure.[11] Moreover, to echo Wright (and Frege and Dummett), these specifications must be "absolutely on the surface", not "tacked on externally". In other words, they must not be *derived* from the DP Axioms or HP; they must be part of the characterizations themselves. Our claim is that neither the DP Axioms nor HP satisfies FC_{tc} under this construal.

3.1 Nominal and Numerical Transitive Counting

In order to argue for this claim, we first need to distinguish between two locally behaviorally indistinguishable kinds of transitive counting, which we call *nominal* and *numerical transitive counting.* Both involve using numerals to answer 'how many'-questions.[12] Moreover, both involve the deployment of a procedure characterized by the following routine:

TC Routine

If asked "How many *F*s are there?"
i. Isolate the *F*s from the non-*F*s.
ii. Establish a bijection between the *F*s and an initial segment of the numerals in the count list <'n_1' . . . 'n_n' > by reciting (possibly non-verbally) the numerals in order, starting with 'n_1' and correlating each *F* with a unique numeral in the list.
iii. If 'n_k' is the final numeral resulting from (ii), then answer "There are n_k *F*s".

In both cases, a condition on *successful* counting is that the k^{th} numeral delivered in step-iii correctly answers the original 'how-many'-question.

Yet there are important differences between the two sorts of counting regarding the extent to which the counter has mastery of numerical concepts. Though we elaborate below, to a first approximation, the idea is that numerical transitive counting requires that one *grasp cardinality concepts* when following the TC Routine. Roughly equivalently, the counter must (tacitly) grasp that the numerals they deploy designate cardinalities.

By contrast, *nominal* transitive counters need not grasp cardinal concepts when deploying numerals in the TC Routine. Of course, the numerals they use may express *our* concepts of natural number, or denote natural numbers in the public language. But the nominal counter needn't grasp this. It suffices that they memorize and follow the TC Routine in a brute mechanical way, as it were, with no appreciation that their numerals designate cardinalities, or that their response designates the cardinality of the collection being counted.

We may now ask whether – and if so in what respects – the neologicist's preferred foundation provides the cognitive or epistemic prerequisites for either kind of transitive counting, where the DP axioms do not. We begin with nominal counting.

3.1.2.1 Prerequisites for nominal transitive counting

One obvious prerequisite for nominal transitive counting is the capacity to *intransitively* count – the numerals in their canonical order. This is required by step-ii of the TC Routine – correlating each *F* with a unique, recited numeral. But what representational resources are required to count intransitively?

First, one needs access to *numerals*. Consider again the DP Novice. Because she has access to the DP Axioms and second-order logic, she can generate a potentially infinite list of numerals. She has a name for the first numeral, and can generate subsequent numerals via the successor axiom, and so is able to generate what we'll call *DP numerals*:

$$< \mathrm{s}(\mathrm{o}), \mathrm{s}(\mathrm{s}(\mathrm{o})), \ldots >$$

What about HP? Consider another fictional character, *the HP Novice*, who grasps HP and second-order logic. Accordingly, he has the resources to form what we call *HP numerals*:[13]

$$< \#[\lambda y.y = \#[\lambda x.\neg x = x]],\ \#[\lambda z.z = \#[\lambda y.y = \#[\lambda x.\neg x = x]] \vee z = \#[\lambda x.\neg x = x]]],\ \ldots >$$

The first HP numeral refers to the number of the concept *being the number of non-self-identical objects*, i.e. the cardinal number one, the second refers to the concept *being the number of the concept being either the number of the number of non-self-identical concepts or the number of self-identical concepts*, i.e. the cardinal number two, etc. Thus, the HP Novice's access to second-order logic provides one important precondition for intransitive counting.

More than just access to numerals is required, however. One must also reliably deploy those numerals in their appropriate *order.* Following Gelman and Gallistel (1978), this is sometimes referred to as the Stable Ordering Principle:

(SOP) The numerals employed in counting must occur in a stable, and thus repeatable, order.

Minimally, this requires that the count sequence contains a fixed first element, followed by a fixed sequence of successive elements. The DP Axioms provide just that, of course: the axiom for zero provides a stable first element, while the axioms characterizing successor provide a sequence of stable successive elements. Not so with HP, however. Though the HP Novice can *form* HP numerals, HP and second-order logic alone do not guarantee the ordering provided. Thus, while the DP Novice plausibly grasps SOP, the HP Novice does not. Further, since SOP is a necessary condition for intransitive counting, it follows that the HP Novice is unable to nominally transitively count.

3.1.1.2 Adjudicating via Nominal Transitive Counting?

Suppose we frame the question differently: *Given an ability to intransitively count*, does FC_{tc} Adjudicate in favor of HP over the DP Axioms? By stipulation, someone is a *nominal* transitive counter only if they can perform the TC Routine correctly *without* grasping that the numerals used in this procedure designate cardinalities. So, the question is: Once provided with the ability to intransitively count, is the HP Novice thereby able to correctly perform the TC Routine whereas the DP Novice is not? The answer, we submit, is "No".

To be a nominal transitive counter one must be able to perform steps i-iii of the TC Routine. Consider each in turn:

Step-i: This consists in isolating the *F*s from the non-*F*s. Here we think *neither* the HP Novice nor DP Novice will be able to perform this task merely by virtue of their grasping second order logic along with HP or the DP axioms. This is hardly surprising, as step-i just is *categorization* – a capacity that is plausibly a precondition, not merely for counting, but almost all cognition. It would be truly remarkable, then, if a foundation for arithmetic provided this prerequisite.

Step-ii: Unlike step-i, this is specific to transitive counting. Assuming both Novices can intransitively count, both can plausibly perform step-ii. After all, both can establish one-to-one correspondences between their numerals and collections of objects, thanks to their access to second-order logic. The only difference will be the kinds of numerals involved.

Step-iii: Assuming nominal transitive counting, step-iii is equivalent to what psychologists call the Last Word Rule [Fuson, 1988]:

(LWR) The last numeral used in the transitive counting procedure answers the relevant 'how many'-question posed when performing that procedure.

Plausibly, neither Novice grasps LWR. For even assuming they can perform step-ii, it is an *extra* item of procedural knowledge – or know how – that they should respond with the final numeral recited in step-ii, as opposed to any other numeral, or no numeral at all.[14]

In sum, neither the DP axioms nor HP provide the cognitive prerequisites for nominal transitive counting, though the DP axioms perform slightly better thanks to satisfying SOP. Hence, FC_{tc} won't Adjudicate in favor of HP if nominal transitive counting is the relevant capacity.

3.1.2.1 Prerequisites for numerical transitive counting

This leads to another possible characterization of FC_{tc}: *Given mastery of the TC Routine*, HP provides the prerequisites for *numerical* transitive counting, whereas alternatives, especially the DP axioms, do not. Put differently, given the TC Routine, whereas the HP Novice can numerically transitively count, the DP Novice cannot.

Recall that numerical transitive counting requires reliably answering 'how-many'-questions via the TC Routine while grasping that the numerals deployed designate the cardinality of the collection being counted. Now, an important difference between numerical and nominal transitive counting concerns step-iii of the TC Routine. In the case of nominal counting, this step is equivalent to LWR, whose grasp does not imply recognition of cardinality. In contrast, *numerical* transitive counting requires formulating step-iii as involving a grasp of what, following Gelman and Gallistel (1978), is known as *the Cardinal Principle*:

(CP) The last numeral used in the TC Routine designates the cardinality of the collection being counted.

Grasping CP requires (tacit) recognition that the terminal numeral deployed in the TC Routine designates a cardinality, or numerosity. As Thompson [2010] explains:

> To be considered to have grasped [CP], a child needs to appreciate that the final number name is different from the earlier ones in that it not only 'names' the final object, signaling the end of the count, but also tells you how many objects have been counted: it indicates what we call the numerosity of the collection.

For example, if the last numeral used in the count is '10', then the counter must (tacitly) recognize that the cardinality of the collection is ten.

This also has ramifications for how we should characterize step-ii of the TC Routine. In particular, since the numerical transitive counter grasps CP and can reliably count collections with different cardinalities, they must also grasp that each successive numeral in the count routine designates a different cardinality. To see this, first note that in order to be a *reliable* numerical transitive counter, one needs to (tacitly) recognize that, had the terminal numeral been any of those prior to 'n_k', the cardinality of the collection would have been different. Start with a collection of ten objects. Had the last numeral used been '9', the counter would need to (tacitly) recognize that the cardinality of the collection would not have been ten. But, by parity of reasoning, the same will be true of any numeral prior to '9' as well. In short, reliable numerical transitive counting appears to require a grasp of something like *successor*.[15,16]

Two further points are worth emphasizing. First, this notion of successor needs to tie the numerals deployed in the TC Routine to cardinalities. Thus, we distinguish two kinds of successor. The first, defined by the DP axioms, is what we call *structural successor*. As we have seen, this notion suffices to generate stably ordered numerals. However, like the rest of the DP axioms, it fails to link numerals to cardinalities. The second notion is *cardinal successor*, as defined by Frege (1884, 1893), given in §1.2 above. Unlike structural successor, cardinal successor directly links concepts to their cardinalities. Thus, it represents a plausible candidate for tying the result of performing the transitive counting procedure to the cardinality of the collection being counted.

Secondly, cardinal successor will not establish a link between transitive counting and cardinalities unless an appropriate interpretation of the initial numeral in the count sequence is provided. Otherwise, nothing prevents the counter from enumerating the first object counted as e.g. 17 rather than 1, thus generating the wrong answer. Hence, reliable numerical counting requires grasping that the first numeral in the count list designates the cardinal number *one*.

With the above considerations in place, we are now in a position to supplement CP by two further conditions on numerical transitive counting:

> (NTC1) The counter must (tacitly) recognize that the first numeral in the count list designates the cardinal number one.

> (NTC2) The counter must (tacitly) recognize that each numeral in the count list following the first designates a cardinal number which is the cardinal successor of some cardinal number designated by a prior numeral.

NTC1 and NTC2 jointly guarantee that the final numeral reached in performing the TC Routine designates the actual cardinality of the collection being counted. Hence, together with CP, these principles plausibly characterize the cognitive or epistemic prerequisites for genuine numerical transitive counting.

3.1.2.2 Adjudication via Numerical Transitive Counting?

Let's now return to the issue of Adjudication. It is at least arguable that the DP Novice has *some* generic notion of number, namely a notion of *structural* number, as defined by the DP axioms. This is adequate for doing basic calculations and proving theorems about these numbers. Moreover, she possesses an accompanying notion of *structural* successor. Still, because she clearly lacks notions of *cardinal* number and *cardinal* successor, she fails all our conditions on numerical counting, NTC1, NTC2, and CP. This reveals that the DP Axioms alone fail to provide the prerequisites for numerical transitive counting, assuming further deduction is not allowed.

But is the HP Novice any better off? Clearly, he *does* possess a notion of cardinal number – or *number of* – thanks to his knowledge of HP. What's more, we are prepared to concede that he satisfies NTC1 – he can see that the first HP numeral refers to a singleton concept, and so can infer via HP that all one-membered classes are numbered by it. However, because he does not yet have a notion of cardinal successor, he does *not* satisfy NTC2. Consider the second HP numeral and a class of two objects, say the moons belonging to Mars. In order to infer from HP that the number of Martian moons is the number referenced by the second HP numeral, he needs to make an additional inference, given the numeral's disjunctive character. Specifically, he needs to infer that the number of the concept *being the number of the concept being non-self-identical* and the number of the concept *being non-self-identical* are distinct. Yet this is *not* something he can know from HP and second-order logic alone, at least not without further deduction. The point generalizes to all HP numerals beyond the first, thus revealing that access to HP and second-order logic alone fails to provide the cognitive prerequisites for numerical transitive counting.

Finally, the previous point implies that the HP Novice lacks the prerequisites for grasping CP. To satisfy CP, the counter must recognize that the last numeral used in the transitive counting procedure designates the cardinality of the collection being counted. Again, this requires recognizing that the last numeral recited in the TC Routine designates a cardinality distinct from those designated by prior numerals in the count list. And this, of course, requires grasping the notion of cardinal successor, which the HP Novice does not have.

In sum, neither Novice has the cognitive prerequisites for numerical transitive counting. Thus, neither HP nor the DP axioms will satisfy Frege's Constraint if numerical transitive counting is the application relevant to its satisfaction. In other words, FC_{tc} will not adjudicate between these as the uniquely correct foundation for arithmetic.

3.2 Recovering Numerical Transitive Counting: Parallel Predicaments

If neither Novice possesses the resources necessary for numerical transitive counting, then what additional resources would each Novice need? What else needs to be added to each set of characterizing principles to satisfy FC_{tc}? Although the HP Novice plausibly recognizes that his first numeral designates the cardinal number one, what he needs, of course, is cardinal *successor*. With this in hand, following Frege, he could go on to prove the existence of zero, that the successor relation is one-to-one on the finite cardinal numbers, and the induction principle. That is, he could establish the DP axioms for cardinal number and successor à la Frege's Theorem. This would provide him with a list of numerals appropriate for numerical transitive counting, generated in a way similar to the DP numerals discussed above.

On the other hand, the DP Novice is missing a general notion of *cardinal* number. Though she can generate numerals and correlate collections, she does not yet know that if a one-to-one correspondence exists between a collection of *F*s and the numerals 'n_1', ..., 'n_k', then the terminal numeral designates the cardinality of the *F*s. This would be remedied if, following Dedekind (1888), she had access to what we call *Dedekind's Theorem*.

> 161. Definition. If Σ is a finite system, then by (160) there exists one and by (120), (33) only one single number n to which a system Z_n similar to the system Σ corresponds; this number n is called the number [Anzahl] of elements contained in Σ (or also the degree of the system Σ) and we say Σ consists of or is a system of n elements, or the number n shows how many elements are contained in Σ. If the numbers are used to express accurately this determinate property of finite systems they are called cardinal numbers.

In contemporary terms, "finite system" translates as "Dedekind finite set", and "similar" as "equinumerous". In effect, Dedekind's Theorem combines previous results—results obtained from the DP axioms, suitable definitions, and second-order logic—to establish the sorts of one-to-one correspondences characteristic of transitive counting, and then defines "cardinal number of the *F*s" as the terminal number resulting from performing the TC Routine on some finite collection of *F*s. With Dedekind's Theorem, the DP Novice is knowingly able to answer 'how many'-questions by using DP numerals which designate cardinalities.

In sum, with additional resources at hand, both Novices would be able to numerically transitively count. The problem, however, is that these additional resources come too late in the explanation, by neologicist lights. They are "tacked on externally" to quote Frege [1903]. They are not "absolutely on the surface" to quote Wright [2000]. More specifically, the problem is that they are *derived* from the characterizing principles, as the passage from Dummett [1991, p. 60], cited in §1.5, makes clear. Again, according to Dummett, because the empirical applications of arithmetic are *essential* to the naturals, those applications must be directly reflected in the principle(s) characterizing those numbers, not subsequently derived from them. Yet what our two Novices reveal is that the cognitive or epistemic prerequisites required for transitive counting are available only if subsequent derivation *is* allowed.

We conclude that even on a construal of Frege's Constraint which permits derivation, HP and the DP axioms are in exactly parallel predicaments. Neither Novice has the resources immediately available for numerical counting, but both would be in the position to do so with some additional resources.[17] In both cases, what's needed are some further definitions and theorems showing that the definitions are adequate to their purpose. The DP Novice has a notion of *structural* number adequate for doing number theory. What she lacks is a notion of *cardinal* number, as provided by Dedekind's Theorem.

In contrast, the HP Novice has a notion of cardinal *number*, as provided by HP. What's missing is an understanding that HP numerals (and the cardinals that they designate) have a structure that is suitable for transitive counting, namely that there is a first such number, that each cardinal number has a successor obtained by adding exactly one object to the previous collection, and that cardinal successor is one-to-one. In short, what is missing is the bulk of Frege's Theorem.

Finally, in both cases, the resources needed to make the transition are remarkably similar. Both Novices would invoke basic facts about one-to-one correspondences, derived via the usual definitions in second-order logic (or elementary set theory). So the two accounts are, from this perspective, on a par.

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Notes

[1] See Heck [2011] for details.

[2] The exact status of HP – especially its epistemic status – is a point of some debate. For present purposes, however, we need not concern ourselves with such matters.

[3] Nor, for that matter, does the development of mathematics itself, with its relentless drive for abstraction, and disconnection from intuitive applications. Frege's main opponents, Weierstrass, Dedekind, and Cantor, all contributed to an emerging trend of divorcing mathematics from Kantian intuition of space and time and, indeed, from any and all applications (see Coffa [1991]). The culmination of this trend was Hilbert's *Grundlagen der Geometrie* [1899]. Of course, Hilbert was aware that spatial intuition or suitably idealized observation – typical applications – remains the source of the axioms of geometry. In his mathematical writing, however, the role of intuition is carefully and rigorously limited to

motivation and heuristic. Once the axioms have been formulated, intuition is banished. From the perspective of the mathematician, *anything at all* can play the role of the undefined primitives of points, lines, planes, etc., so long as the axioms are satisfied. Otto Blumenthal reports that in a discussion in a Berlin train station in 1891, Hilbert said that in a proper axiomatization of geometry, "one must always be able to say, instead of 'points, straight lines, and planes', 'tables, chairs, and beer mugs'" (see "*Lebensgeschichte*" in Hilbert [1935, 388-429]; the story is related on p. 403). Presumably, the same goes for number theory. There are, of course, important questions concerning which mathematical theory is best applied to this or that part of material (or non-material) reality. What, for example, is the structure of physical space? But, for Hilbert, these applications are, *and should be*, "tacked on externally". Arguably, this orientation has dominated mathematics ever since.

[4] There is a parallel issue concerning the proper interpretation of Frege's own use of Frege's Constraint when it comes to real analysis. We need a reading which (i) is reasonably plausible as a condition to impose on an account of the real numbers and (ii) is not met by the celebrated accounts of Cantor and Dedekind. We cannot find such a reading.

[5] See Samuels, Shapiro, and Snyder [ms.].

[6] To illustrate, although it could be essential to possessing the concept of water that we can recognize water, there is no reason to suppose that it is essential to water, as such, that it can be so recognized.

[7] For example, is it really that obvious that someone who could prove complex results in number theory would lack natural number concepts merely because they cannot count transitively?

[8] For example, it is not knowledge of the empirical regularity that when one takes three things and adds four more things, one gets seven things (at least for a short period of time).

[9] ibid, emphasis added.

[10] See, for example, Carey [2009], especially Chapter 8.

[11] There is another possible construal of "building in transitive counting" on which a foundation for arithmetic builds in transitive counting when it specifies the conditions under which *successful* transitive counting occurs. However, since this success conditions reading has the very same problems as the present view, we relegate our discussion of it to footnotes.

[12] Although we tend to talk as if numerals are elements of a symbol system, such Arabic notion, other things – such as, natural language expressions, manual signs, or even body parts – would suffice.

[13] Many thanks to Neil Tennant here.

[14] In fact, during the process of learning to transitively count, children appear to go through a period when they perform step-ii, yet fail to recognize that 'n_k' is the appropriate answer to the initial 'how many'-question.

[15] Incidentally, this is a point recognized by Sarnecka and Carey [2008], who note that "...knowing the cardinal principle means having some implicit knowledge of the successor function – some understanding that the cardinality for each numeral is generated by adding one to the cardinality for the previous numeral."

[16] Strictly speaking, the present argument does not show that any numerical counter requires the notion of successor. In particular, if a creature has only a finite set of numerals – say, '1, 2 . . . 10' – then the argument only requires that they grasp the analog of successor up to the limit imposed by their count list. However, since both Novices can be given numerals for all finite positive integers, for them grasping CP requires grasping successor.

[17] An anonymous referee queries whether all of Frege's Theorem is needed for the HP Novice to transitively count, or whether Hume's Principle along with certain additional facts like the number of *F*s is 3 would suffice. Admittedly, the HP Novice can answer some "how many" questions if she is supplied with names for particular cardinal numbers. Suppose, for example, that we tell the HP Novice that zero is the number of the concepts of being non-self-identical, that one is the number of the concept of being identical to zero (following Frege so far), that two is the number of the concept of either being identical to zero or identical to one, and that five is the number of fingers on her left hand. Then she can answer 'How many?'-questions, provided that the answer is among the set {zero, one, two, five}. She can say, for example, how many fingers are on her right hand and that two is the number of parents she has. But this is a rather thin range of application. Someone who has grasped transitive counting can, in principle, answer 'how many'-questions for any finite collection (or at least any collection up to the limit of her count sequence).

The same referee asks if there is a way to compare the strength of the additional resources that the HP Novice and the DP Novice need to transitively count. In effect, the question concerns the resources needed to prove Frege's Theorem and what we call "Dedekind's Theorem". Technically, the questions are subtle, and most interesting. Frege's Theorem has been well-studied in this regard. The upshot is that to derive the Dedekind-Peano axioms from Hume's Principle, one needs at least some impredicative comprehension, something in the neighborhood of Δ^1_1-comprehension, plus some impredicativity in Hume's Principle itself. And the particular impredicativity assumed in the logic will affect the impredicativity of induction in the resulting arithmetic. See, for example, Burgess [2005] or Heck [2011]. There is no similar study of Dedekind's theorem; Dedekind himself does not impose any restrictions on induction, but the sources and targets in both cases are similar. We will not speculate on the relevance of these results for the philosophical programs. Suppose, for example, that Frege's Theorem requires slightly less impredicativity than Dedekind's Theorem. Would that give an advantage to abstractionism over structuralism?

Finally, the referee suggests the following possibility: instead of transitive counting, perhaps the application relevant to satisfying Frege's Constraint is cardinality comparison. After all, there is a sense in which the HP Novice can compare cardinalities in a way that the DP Novice cannot: given any two sets, only the HP Novice can tell whether or not they have the same number. Thus, abstractionism may have an advantage after all.

Note first that 'cardinality comparison' is ambiguous: it could connote sameness and difference in cardinality, or it could connote having a greater or lesser cardinality. While the HP Novice can compare cardinalities in the first sense, neither Novice can compare cardinalities in the second. For example, if there are two Elmos and three Grovers, neither Novice knows that that there are more Grovers than Elmos. Insofar as it may seem independently plausible that grasping natural number concepts requires an ability to compare cardinalities, we surmise that it would be so in virtue of this *second,* richer sense. On the other hand, there is no evident reason for thinking that 'cardinality comparison' in the first, rather thin sense is the *primary* application of the naturals. Moreover, the stronger, relevant principle for cardinality comparison is not an abstraction principle.

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