Generating Functions Associated to Species

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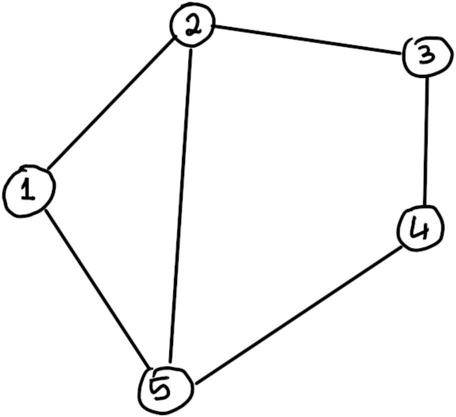
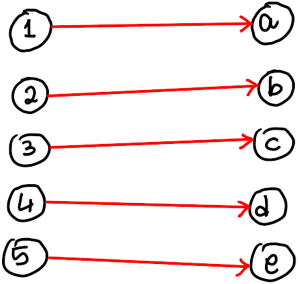
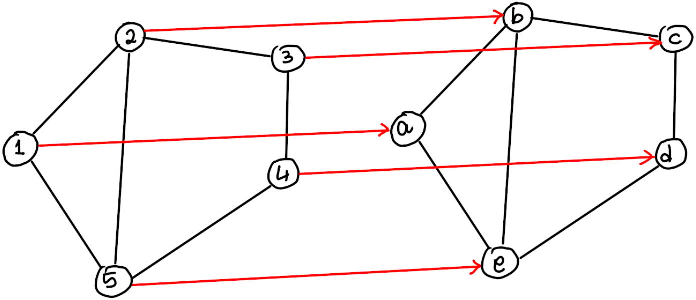
A structure labeled by $\{1, 2, 3, 4, 5\}$



A bijection $\{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$



Transport of Structure



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A species is a functor $F : \mathrm{FB} \to \mathrm{FB}$.

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Transport of structure

If $f : U \to V$ is a bijection, then,

$$\sigma : u \to u',$$

if and only if

$$\mathcal{S}[f](\sigma) : f(u) \to f(u').$$

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Given a species F, its generating series is the formal power series:

$$F(z) = \sum_{n \geq 0} \frac{|F[n]|}{n!} z^n.$$

For example,

$$\mathcal{G}(z) = \sum_{n \geq 0} \frac{2^{\binom{n}{2}}}{n!} z^n,$$

$$\mathcal{A}(z) = \sum_{n \geq 0} \frac{n^{n-2}}{n!} z^n,$$

$$\mathcal{G}^c(z) = \log \mathcal{G}(z),$$

$$\mathrm{Par}(z) = \exp(\exp(z) - 1),$$

$$\mathcal{S}(z) = L(z) = \frac{1}{1 - z}.$$

Isomorphism of Species

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An isomorphism $\phi : F \tilde{\rightarrow} G$ of species is a collection of bijections

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that respects transport of structure, i.e., for every bijection $f : U \to V$, the diagram

$$\begin{tikzcd}
F[U] \arrow[r, "F[f]"] \arrow[d, "\phi_U"'] & F[V] \arrow[d, "\phi_V"] \\
G[U] \arrow[r, "G[f]"'] & G[V]
\end{tikzcd}$$

commutes.

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$$\mathcal{S}(z) = L(z), \text{ but } \mathcal{S} \text{ and } L \text{ are not isomorphic.}$$

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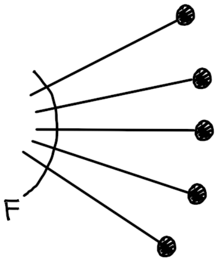
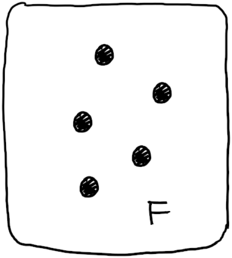
Visualization on an F-structure on U





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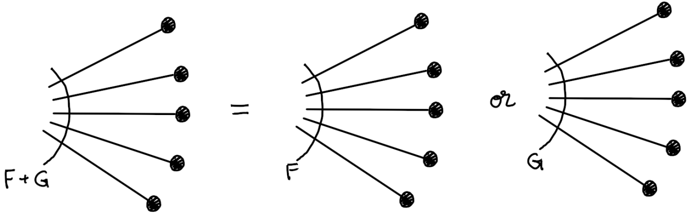

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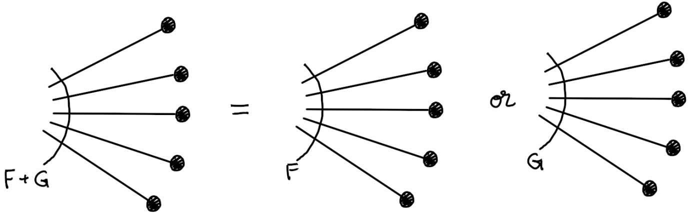

$$(F + G)(z) = F(z) + G(z).$$

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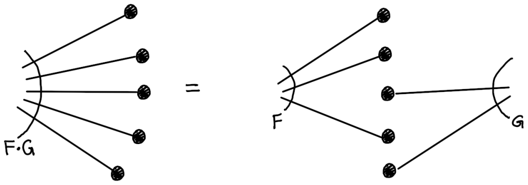

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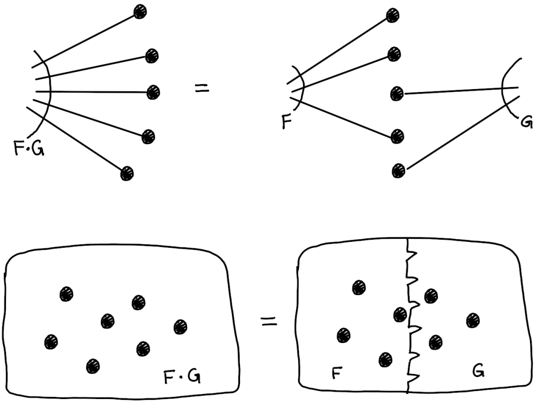

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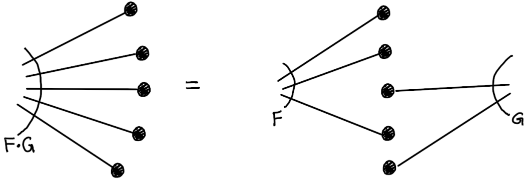

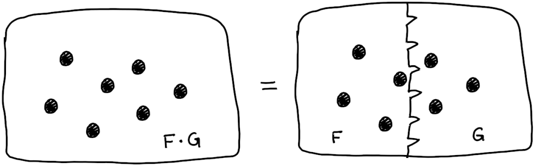

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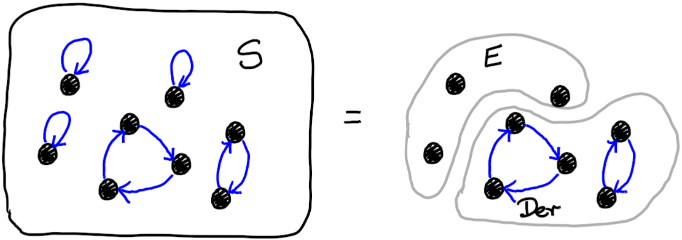

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$$\lim_{n\to\infty} \frac{|\mathrm{Der}[n]|}{|\mathcal{S}[n]|} = 1/e.$$

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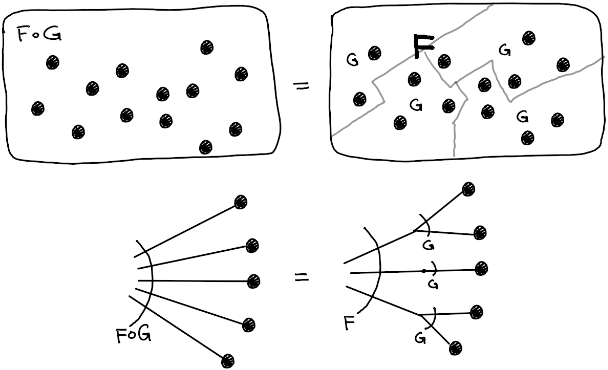

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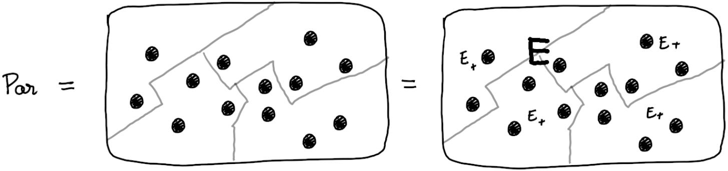

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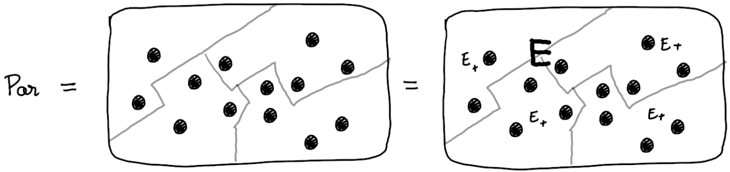

$$\mathrm{Par}(z) = \exp(\exp(z) - 1),$$

The exponential generating function for *Bell numbers*.

The Cycle Index Generating Function

$$Z_F(x_1, x_2, \dots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} |\mathrm{Fix}(F[n]; \sigma)| x_1^{m_1(\sigma)} x_2^{m_2(\sigma)} \cdots ,$$

where $m_i(\sigma)$ is the number of i-cycles in the permutation σ, and $\mathrm{Fix}(F[n]; \sigma)$ denotes the number of F-structures on $[n]$ that are fixed by σ, a permutation of $[n]$.

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- $Z_F(z, 0, 0, \dots) = F(z)$.
- $Z_F(z, z^2, \dots) = \sum_{n \geq 0} |S_n \backslash F[n]| z^n$ (Burnside's lemma).

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$$\begin{aligned} Z_{F \circ G}(x_1, x_2, \dots) &= Z_F(Z_G(x_1, x_2, \dots), Z_G(x_2, x_4, \dots), Z_F(x_3, x_6, \dots)) \\ &= (Z_F \circ Z_G)(x_1, x_2, \dots). \end{aligned}$$

Symmetric Functions

Power Sum

$$p_k = \sum_{i \geq 1} t_i^k.$$

Complete

$$h_k = \sum_{i_1 \leq \cdots \leq i_k} t_{i_1} \cdots t_{i_k}.$$

Elementary

$$h_k = \sum_{i_1 < \cdots < i_k} t_{i_1} \cdots t_{i_k}.$$

Bases of Symmetric Functions

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Then $\{p_\lambda\}$, $\{h_\lambda\}$, and $\{e_\lambda\}$ form bases of the space of all symmetric functions.

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Note that $\mathrm{ch}_n V$ is a homogeneous symmetric function of degree n.

Theorem (Frobenius)

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The fact that irreducible characters form a basis of class functions of S_n is manifested in the fact that $\{s_\lambda\}$ form a basis of symmetric functions.

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$$\mathrm{ch}_n\, \mathbb{C}[X] = \frac{1}{n!} \sum_{\sigma \in S_n} |\mathrm{Fix}(X;\sigma)| p_1^{m_1(\sigma)} p_2^{m_2(\sigma)} \cdots .$$

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We have:

$$\mathrm{ch}\, F = Z_F(p_1, p_2, \dots).$$

Thus $\mathrm{ch}\, F$ and Z_F carry the same information.

Examples of Total Frobenius Characteristic

$$\operatorname{ch} E = \sum_{n=1}^{\infty} h_n.$$

$$\operatorname{ch} S = \prod_{i=1}^{\infty} \frac{1}{1 - p_i}.$$

$$\operatorname{ch} L = \frac{1}{1 - p_1}.$$

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Corollary

Rows of the character table of S_n have non-negative sum.

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$$h^{-1} = \left(\prod_{i=1}^{\infty} \frac{1}{1 - t_i} \right)^{-1} = \prod_{i=1}^{\infty} (1 - t_i) = \sum_{i=0}^{\infty} (-1)^i e_i.$$

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Let $e = \sum_{i=0}^{\infty}(-1)^i e_i$. Therefore,

$$\mathsf{ch}\,\mathrm{Der} = \sum_{\lambda} e s_\lambda \sum_{|\alpha|=|\lambda|} \chi_\lambda(\alpha).$$

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Therefore,

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We conclude that, for every partition μ,

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$$p(n) - p(n-1) \geq 0,$$

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$$\sum_{k=0}^{n}(-1)^{n-k}p^{\text{self-conjugate}}(k) \geq 0,$$

for all $n \geq 0$.

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It turns out that plethysm of symmetric functions is also related to plethysm of species:

$$\operatorname{ch} F \circ G = \operatorname{ch} F \circ \operatorname{ch} G.$$

Example

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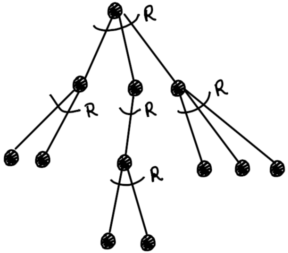
The identity

$$\mathrm{Par} = E \circ (E - 1)$$

implies:

$$\mathsf{ch}\,\mathrm{Par} = h \circ (h - 1).$$

R-enriched tree



Suppose $f = \frac{p_1}{\mathrm{ch} R}$ for some species R

then $f^{[-1]} = \mathrm{ch}\, \mathcal{A}_R$