Nārāyaṇa Paṇḍita's *Turagagati* Method for the Construction of 4x4 Pandiagonal Magic Squares

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Introduction to magic squares

- Toying with magic squares is indeed positively recreational and is known to have fascinated even the greatest of mathematicians.[1]
- A "normal" magic square of order n is an arrangement of n^2 different numbers in a $n \times n$ square array such that the sum of the numbers along every row, column, and leading diagonals is the same.
- If the broken (or wrap-around) diagonals (*alpaśruti*) of the magic square also add up to the magic sum, then the square is called a pandiagonal magic square.

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

Normal magic square (S = 34)

$(7+5+6+8 \neq 34)$

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

Pandiagonal magic square (S = 34)

$(4+16+13+1 = 34)$

Magic squares in India: Its purpose & earliest occurrence

- In the Indian tradition, it is held that magic squares were first taught by Lord Śiva to Maṇibhadra. Nārāyaṇa Paṇḍita's *Gaṇitakaumudī* notes:

> अथ भुवनत्रयगुरुणा उपदिष्टमीशेन माणिभद्राय । (origin of the study)
> कौतुकिने भूताय श्रेढीसम्बन्धि सद्गणितम् ॥
>
> सद्गणितचमत्कृतये यन्त्रविदां प्रीतये कुगणकानाम्। (its three-fold purpose)
> गर्वक्षित्यै वक्ष्ये तत्सारं भद्रगणिताख्यम्॥
>
> 1. In order to embellish the practice of good mathematics
> 2. For pleasing those who are involved with the [construction of] *yantras*.
> 3. For eradicating the arrogance of the impostors.

- The earliest literary evidence for the occurrence of magic squares is to be found in the work ascribed to the famous Buddhist philosopher Nāgārjuna (1st century CE).

The pandiagonal magic square attributed to *Nāgārjuna*

- In his *Kakṣapuṭa-tantra*, Nāgārjuna (100 CE) gives rules for the construction of 4 × 4 squares with even as well as odd sums. These rules are based on an interesting mnemonic expressed in *Kaṭapayādi* notation. A particular case of 4 × 4 square with the magic sum 100 is presented in the verse below:

नीलं चापि दयाचलो नवभुवं खारीवरं रागिनं
भूपो नारिवगो जरा चरनिभं तानं शतं योजयेत् ॥
भूतप्रेतपिशाचराक्षसमुखान् सर्पान् खलान् संहरत्
अग्निं चौरभयादिनाशनमिदं नागार्जुनं निर्मितम् ॥

- The effect of seeing such a square described in the latter half of the verse quite interesting.
- This magic square has been called the *Nāgārjunam*.

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

Nāgārjuna's ***Bhadram*** (S = 100)

This square is formed of four arithmetic sequences namely:

{6, 10, 14, 18}, {16, 20, 24, 28}, {22, 26, 30, 34}, {32, 36, 40, 44}

- Varāhamihira's ***Bṛhat-saṃhitā*** (6th century CE), in Chapter 76, verses 23–26, gives the method of preparing perfumes employing the *sarvatobhadra*.

द्वित्रिइन्द्रियअष्टभागैः अगुरुः पत्रं तुरुष्कयौलेयौ।
विषयअष्टपक्षदहनाः प्रियङ्गमुस्तारसाः केशः॥
स्पृक्कात्वक्तगराणां मांस्याश्च कृतएकसप्तषड्भागाः।
सप्तऋतुवेदचन्द्रैः मलयनखश्रीककुन्दुरुकाः॥

षोडशके कच्छपुटे यथा तथा मिश्रिते चतुर्द्रव्ये।
येऽष्टादश भागास्तेऽस्मिन् गन्धादयो योगाः॥
नखतगरतुरुष्कयुता जातीकर्पूरमृगकृतोब्द्रोधाः ।
गुडनखधूप्या गन्धाः कर्तव्याः सर्वतोभद्राः॥

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

Varāhamihira's *sarvatobhadra* (S = 18)

Sir George A. Grierson on the antiquity of magic squares in India

Sir George Abraham Grierson (1851 – 1941), an Irish administrator in British India, with a keen interest in linguistics pursued studies in Indian languages and literature during his postings in Bengal and Bihar since 1873. In a short article titled "American Puzzle", he notes:[2]

AN AMERICAN PUZZLE.

About seven months ago, tbe *Pioneer*, in a letter headed "From All About," proposes a problem, called tbo "American Puzzle," tbe attempted solution of which is said to have driven sovoral people nearly mad. The problem is to arrango the sixteen consecutivo numbers from 1 to 16, in four rows of four each in such a way that the total of every line and group of four will amount to exactly thirty-four. Tho puzzlo admits of soveral answers, and one is—

Correspondence in Indian Antiquary Vol. 10 - 1881

George A. Grierson, Madhubani, Darbhanga

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

In tho above gronp overy line of fonr, every possible group of four forming a square, and tho sum of tho four corner numbers amounts to 34.

The problem is, however, by no means a modern one, dating, as it does, far back into the history of Indian Astrology. To prove what I say, I append the following extract from the *Jyotis-tattwa* :—

From 1898, Grierson conducted the Linguistic Survey of India (published 1903–28), obtaining information on 364 languages and dialects.

In his text titled *Smṛtitattva*, Raghunanada Bhaṭṭācārya gives the following verses which have been extracted by Grierson:

पञ्चरेखा समुल्लिख्य तिर्यगूर्द्ध्वक्रमेण हि ।
पदानि षोडशापाद्य त्वेकमाद्ये मुनौ त्रयम् ॥
नवमे सप्त दद्यात्तु बाणं पञ्चदशे तथा ।
द्वितीयेऽष्टावष्टमे षट् दिशि द्वौ षोडशे श्रुतिः ॥

1	8		
		3	6
7	2		
		5	4

⇝

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

Having drawn five lines horizontally and vertically, and thereby creating sixteen cells, in place 1 in the first of those, 3 in the seventh 3, 7 in the ninth, 5 in the fifteenth, 8 in the second , 6 in the eighth, 2 in the tenth, and 4 in the sixteenth.

Having explained how to construct the basic framework it is said:

एकादिना समं ज्ञेयमिच्छाङ्कार्धं त्रिकोणके ।
तदा द्वात्रिंशदादिः स्यात् चतुष्कोष्ठेषु सर्वतः ॥ (सर्वतः – any which way you choose!)

Effects of possessing/having sight of magic squares

दर्शनाद्धारणात् तासां शुभं स्यात् एषु कर्मसु । द्वात्रिंशत् प्रसवे नार्याःचतुस्त्रिंशद्गमे नृणाम् ।
भूताविष्टेषु पञ्चाशत् मृतापत्यासु वै शतम् । द्वासप्ततिस्तु वन्ध्यायां चतुःषष्टी रणाद्धनि ।
विषे विंशो धान्यकीटेष्वष्टाविंशतिरेव च । चतुरष्टौ च बालानां रोदने परिकीर्तिता ॥

Magic sum	Effect of seeing such magic squares
32	Useful to a woman in childbirth
34	Used when setting out on a journey
50	Used for casting out devils
100	Used for women whose children have died
72	Used for a barren woman
64	Used in the tumult of battle
20	Used in cases of poisoning
28	Used when paddy is attacked by insects
84	Used for hushing children when they are crying

Manuscript of *Gaṇitakaumudī*

- The focus of this talk is to present the algorithm for constructing pandiagonal magic squares of order 4 by *Turagagati* method as propounded by Nārāyaṇa Paṇḍita in his *Gaṇitakaumudī*.
- In recent times, Henry Thomas Colebrooke, identified the presence of this text (Library of the India Office).
- The very first edition of the full text of *Gaṇitakaumudī* was by Pandit Padmākara Dvivedi, brought out as a two part publication in the years 1936 and 1942.
- Kusuba (1993) has also brought out the edited and transliterated version of the last two chapters. He also correctly constructed the 24 possible configurations of magic squares that Nārāyaṇa himself has alluded in the text, which seems to have errors in the edition of Dvivedi.
- Paramanand Singh has brought out the translation of the various chapters *Gaṇitakaumudī* as a series of publications since 1998.
- This work needs a thoroughly revised edition, as the current one is not satisfactory.

Earlier related studies

- Lehmer (1933) surveyed 4 x 4 squares and concluded that there are 539,136 semi magic squares, 7,040 normal magic squares and only 48 pandiagonal magic squares are possible.
- Later, Rosser and Walker (1938) have corrected this result and have mathematically arrived at the conclusion that there are 384 pandiagonal squares (as stated by Nārāyaṇa Paṇḍita).
- Vijayaraghavan (1942) in his paper on Jaina Magic Squares deals with 4 x 4 pandiagonal magic squares, provides a mathematical analysis, brings out various properties.
- The article of Datta and Singh revised by K.S Shukla (1992), as well as the writings of R.C. Gupta (2005) have presented this method briefly.
- Bhowmik (2018) has worked on the proofs demonstrating certain properties.
- In all these works, the exact algorithm prescribed by Nārāyaṇa Paṇḍita for constructing magic squares with *Turagagati* has not been satisfactorily described and analysed in detail.

Brief summary of the contents in *Gaṇitakaumudī*

Number	Chapter Title	Mathematical topics covered
1	*Prakīrṇaka-vyavahāra*	Logistics, weights and measures
2	*Miśraka-vyavahāra*	Partnership, sales, interest, etc.
3	*Śreḍhī-vyavahāra*	Sequences and series
4	*Kṣetra-vyavahāra*	Geometry of planar figures
5	*Khāta-vyavahāra*	Excavations
6	*Citi-vyavahāra*	Stacks
7	*Rāśi-vyavahāra*	Mounds of grain
8	*Chāyā-vyavahāra*	Shadow problems
9	*Kuṭṭaka*	Linear indeterminate equations
10	*Vargaprakṛti*	Quadratic indeterminate equations
11	*Bhāgādāna*	Factorisation
12	*Rūpādyamśāvatāra*	Partitioning unity into unit-fractions
13	*Aṅkapāśa*	Combinatorics
14	*Bhadragaṇita*	Magic squares

Gaṇitakaumudī's date of composition

- The date of composition of *Gaṇitakaumudī* has been given by Nārāyaṇa Paṇḍita himself in the following verse that appears towards the end of the text.

  गजनगरविमितशाके दुर्मुखवर्षे च बाहुले मासि ।
  धातृतिथौ कृष्णदले गुरौ समाप्तिं गतं गणितम् ॥

  The *Gaṇita* (*Gaṇitakaumudī*) came to completion on Thursday, 2nd *Tithi* of the *Kṛṣṇa Pakṣa* (waning cycle of Moon), of the month *Kārtika* in *Durmukha Saṃvatsarā*, in *Śaka* 1278 (*gaja* - 8; *naga* - 7; *ravi* - 12).

- Thus we unambiguously know that the work *Gaṇitakaumudī* got completed in the year **1356 CE (1278 *Śaka* year)**. It is interesting to note that Nārāyaṇa Paṇḍita has specified the *tithi* count in terms of the *devatā* of the *tithi*. Based on the list available from other works, we know that *dhātṛtithi* corresponds to *dvitīyā*.

Nārāyaṇa Paṇḍita's ode to his father

- Unfortunately, not much is known about the Nārāyaṇa Paṇḍita's date of birth, whereabouts and other biographical details. One verse in *sragdharā* meter that has been dedicated to capture the glory of his father is as follows:

आसीत् सौजन्यदुग्धांबुधिरवनिसुरश्रेणिमुख्यो जगत्यां
प्रख्यः श्रीकण्ठपादद्वयनिहितमनाः शारदाया निवासः ।
श्रौतस्मार्तार्थवेत्ता सकलगुणनिधिः शिल्पविद्याप्रगल्भः
शास्त्रे शस्त्रे च तर्के प्रचुरतरगतिः श्रीनृसिंहो नृसिंहः ॥

He was the milky ocean of nobility (*saujanya*), the foremost in the assembly of *brahmaṇas* (*avanisura*) whose fame has spread over the world; [He was] one whose mind was steadfast (*nihita*) at the feet of Lord Śiva; one who was the dwelling place of Devī Sarasvatī; one who had mastered the [performances of] *śrauta* and *smarta* [*karmas*]; one who was a reservoir of all virtues; one who was outstanding in the field of architecture/geometry; one who had great felicity (*pracurataragati*) in *śāstrās*, rituals and logic; [my father] by name Śri Nṛsiṃha was indeed a *nṛsiṃha* (lion among men).

Nārāyaṇa Paṇḍita's classification of magic squares

- One of the notable features of Nārāyaṇa Paṇḍita is that he methodically introduces all topics that he discusses.
- For instance, after setting out the historical background and context, he commences this chapter by broadly classifying three types of magic squares.

> समगर्भविषमगर्भे विषमञ्चेति त्रिधा भवेद् भद्रम्।...
> भद्राङ्के चतुराप्ते निरग्रके तद्भवेच्च समगर्भम्।
> द्व्यग्रे तु विषमगर्भं त्र्येकाग्रे केवलं विषमम्॥
>
> ***Samagarbha***, ***viṣamagarbha*** and ***viṣama*** are the three forms of magic square. When the order of the magic square is divided by 4, if the remainder is **zero**, then it is *samagarbha*; if remainder is **two**, then it is *viṣamagarbha*; and if remainder is **three** or **one**, then it is *viṣama*.

Nārāyaṇa Paṇḍita's *turagagati* method for constructing 4 x 4 pandiagonal magic squares

Verses on constructing of 4x4 magic square by *turagagati*

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ ।

न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च तौ ॥१०॥

सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः ।

समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम् ॥ ११ ॥

तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानाञ्च कर्णगानाञ्च ।

अङ्कानां संयोगः पृथक्पृथङ्क्तितो जायते तुल्यः ॥ १२ ॥

द्वौ द्वौ श्रेढीसमुद्भवौ अङ्कौ (चित्वा)	Having chosen pairs of numbers generated in an arithmetic sequence (*śreḍhi*)
क्रमोत्क्रमेण	in sequence and out of sequence
तौ न्यस्य	and placing those pairs
चतुरङ्गतुरगगत्या	by making horse moves

Verses on constructing of 4x4 magic square by *turagagati*

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ ।
न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च तौ ॥१०॥
सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः ।
समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम् ॥ ११ ॥
तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानाञ्च कर्णगानाञ्च ।
अङ्कानां संयोगः पृथङ्क्तितो जायते तुल्यः ॥ १२ ॥

कोष्ठैक्य	[such that the relative positions of the numbers placed by horse moves from any given cell] are positioned in adjacent cells [along diagonals]
एकान्तरेण च	or at an interval of one cell [either along a row, or column]
सव्यासव्यतुरङ्गम-रीत्या	and by making use of the motion of horse to the left and right

Verses on constructing of 4x4 magic square by *turagagati*

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ ।

न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च तौ ॥१०॥

सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः ।

समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम् ॥ ११ ॥

तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानाञ्च कर्णगानाञ्च ।

अङ्कानां संयोगः पृथङ्गितो जायते तुल्यः ॥ १२ ॥

समगर्भे षोडशगृहभद्रे	in a magic square with 16 cells and of the type 4n
कोष्ठान् प्रपूरयेदङ्कैः	may you fill [all] the cells with the numbers [of the chosen arithmetic sequence]
अयम् विधिः प्रोक्तः	this is the method that has been stated [by earlier mathematicians/himself?]

Verses on constructing of 4x4 magic square by *turagagati*

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ ।
न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च तौ ॥१०॥
सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः ।
समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम् ॥ ११ ॥
तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानाञ्च कर्णगानाञ्च ।
अङ्कानां संयोगः पृथक्ङ्गितो जायते तुल्यः ॥ १२ ॥

तिर्यक्कोष्ठगतानां अङ्कानां संयोगः	[thereby] the sum of the numbers along the rows (*tiryakkoṣṭha*)
ऊर्ध्वस्थानाञ्च	and of those along the column
कर्णगानाञ्च	and of those along the diagonals [including broken diagonals]
पृथक्ङ्गितो जायते तुल्यः	when counted separately will be equal

- After presenting the algorithm in the verses above, Nārāyaṇa Paṇḍita also tabulates 24 different half-filled configurations of the pandiagonal magic squares for a fixed position of 1 in the top-left corner. The table is accompanied with the following brief explanation in prose.

> प्रथमयमलाङ्कयुगलम् १।२।३।४ द्वितीयम् ५।६।७।८ तृतीयम् ९।१०।११।१२ चतुर्थम् १३|१४।१५।१६। प्रथमकोणलग्नैः प्रथमयमलयुगाङ्कैः जाताश्चतुर्विंशतिभेदाः। तेषां दर्शनम्। एवमन्यैर्यमलयुगाङ्कैः पृथक् पृथक् चतुर्विंशतिभेदा भवन्ति ।

- Then it is stated by Nārāyaṇa Paṇḍita —

> एवं चतुर्भद्रस्य चतुर्भिः यमलैः चतुरशीत्यधिक-शत्रयभेदा भवन्ति।।
>
> Thus with just four pairs of numbers (*caturbhiḥ yamalaiḥ*), there are 384 variants of a 4x4 [pandiagonal] magic square.

Notations and sets considered

We consider the following, in order to describe the algorithm.

- Let M be a pandiagonal magic square with 16 cells (*koṣṭhas*) where the cells are denoted by :

$$M_{ij} \quad (i, j = 1, 2, 3, 4)$$

- Let S be the arithmetic sequence (*śreḍhi*) with which the cells of M are to be filled. The sequence S we choose for demonstration is:

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$$

- Subsets have to be conceived from this set of numbers as described by Nārāyaṇa Paṇḍita himself in his commentary as *yamalāṅkayugalam*, meaning pair of pairs. They are:

$$\mathrm{S}_1 = \{1, 2, 3, 4\}, \quad \mathrm{S}_2 = \{5, 6, 7, 8\}, \mathrm{S}_3 = \{9, 10, 11, 12\}, \quad \mathrm{S}_4 = \{13, 14, 15, 16\}$$

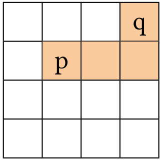
- The horse moves that will be considered for obtaining the magic squares, are represented:

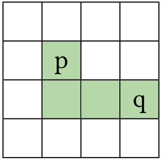
$$H_1 = \{(1, 2), (1, 3), (3, 4)\}, H_2 = \{(5, 6), (5, 7), (7, 8)\},$$

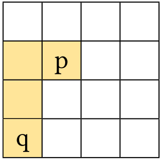
$$H_3 = \{(9, 10), (9, 11), (11, 12)\}, H_4 = \{(13, 14), (13, 15), (15, 16)\}$$

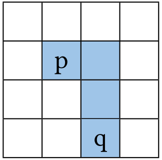
Notations and sets considered

In a 4 x 4 magic square, from any given cell, there are only four types of horse moves that are possible. They are represented by U, D, L and R and illustrated below:







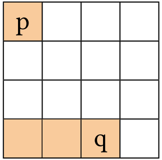


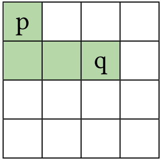
Ūrdhva (up) - move U : $(M_{i,j} \longrightarrow M_{i-1,j+2})$

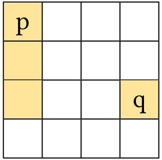
Adhaḥ (down) - move D: $(M_{i,j} \longrightarrow M_{i+1,j+2})$

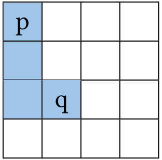
Savya (left) - move L : $(M_{i,j} \longrightarrow M_{i+2,j-1})$

Asavya (right) - move R: $(M_{i,j} \longrightarrow M_{i+2,j+1})$









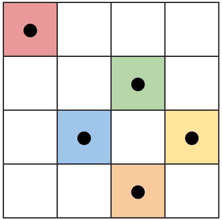
Significance of the phrases *koṣṭhaikya* and *koṣṭhāntara*

- The term *koṣṭhā* employed in the context of magic squares represents a cell. Thus the two terms as such mean the following:

  a. *Koṣṭhaikya* - placement of numbers in adjacent cells
  b. *Koṣṭhāntara* - placement by skipping a cell in-between

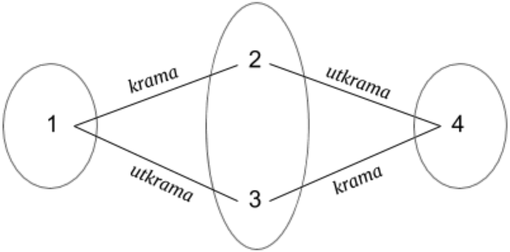
- The significance of these terms is to capture the relative position of all the numbers, that are positioned by the four possible horse moves from a given cell.

- That is, the numbers positioned by horse moves from a given cell get placed in such a way that they will be either in *koṣṭhaikya* or *koṣṭhāntara*.



Significance of the phrase *kramotkrama*

- *Kramotkrama* is a qualifier that can either be attributed to the horse moves (*turagagati*) or to the choice of the numbers paired for making the horse move. In the algorithm that we describe below we *kramotkrama* is a qualifier to the way the pairs are chosen.
- Literally, the phrase *kramotkrama* means along an order (*krama*) or along a different order (*utkrama*). As presented earlier, the arithmetic series is structured as pair of pairs termed *yamalāṅkayugala* by Nārāyaṇa Paṇḍita.
- The pairs of numbers chosen within these sets are in sequence and out of sequence. For instance in S_1, the horse moves listed earlier are (1,2) in *krama* and then (1,3) in *utkrama*.



Rule-based algorithm with only horse moves

Rules for placing pairs in S_1

R.1 The first element 1 of *prathamayamalāṅkayugalam* S_1, is to be placed in any of the sixteen cells.

1			

Rule-based algorithm with only horse moves

Rules for placing pairs in S_1

- R.1 The first element 1 of *prathamayamalāṅkayugalam* S_1, is to be placed in any of the sixteen cells.
- R.2 For a fixed position of 1, 2 can be placed by any one of the four valid horse moves — D / U / L / R, described earlier.

1			
		2	

Rule-based algorithm with only horse moves

Rules for placing pairs in S_1

R.1 The first element 1 of *prathamayamalāṅkayugalam* S_1, is to be placed in any of the sixteen cells.

R.2 For a fixed position of 1, 2 can be placed by any one of the four valid horse moves — D / U / L / R, described earlier.

R.3 Having placed 1 and 2, 3 is also to be placed in a horse move with respect to 1 through any one of the three possible horse moves.

1			
		2	
		3	

Rule-based algorithm with only horse moves

Rules for placing pairs in S_1

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R.2 For a fixed position of 1, 2 can be placed by any one of the four valid horse moves — D / U / L / R, described earlier.

R.3 Having placed 1 and 2, 3 is also to be placed in a horse move with respect to 1 through any one of the three possible horse moves.

R.4 Having placed 1, 2 and 3, 4 is placed such that it is in a horse move from both 2 and 3. There is only one such position for any given placement of 1, 2 and 3.

1			
		2	
4			
		3	

Rules for placing pairs in S_2, S_3 and S_4

R.5 Having placed all elements in S_1, 5 is placed such that it is always in a horse move from 1. There are only two possible ways to place 5, since 2 and 3 are already positioned through horse moves from 1.

1			
		2	
4	5		
		3	

Rules for placing pairs in S_2, S_3 and S_4

R.5 Having placed all elements in S_1, 5 is placed such that it is always in a horse move from 1. There are only two possible ways to place 5, since 2 and 3 are already positioned through horse moves from 1.

R.6 All numbers in each of the *yamalāṅkayugalams* of S_2, S_3 and S_4 get placed by choosing the same set of horse moves chosen for the pairs in *prathamayamalāṅkayugalam* S_1, with the only condition that if a cell is already filled then the horse moves get reversed.

Equivalent pairs horse moves			
H_1	**H_2**	**H_3**	**H_4**
(1,2)	(5,6)	(9,10)	(13,14)
(1,3)	(5,7)	(9,11)	(13,15)
(3,4)	(7,8)	(11,12)	(15,16)

1			
		2	
4	5		
		3	6

R.5 Having placed all elements in S_1, 5 is placed such that it is always in a horse move from 1. There are only two possible ways to place 5, since 2 and 3 are already positioned through horse moves from 1.

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(3,4)	(7,8)	(11,12)	(15,16)

1			
		2	7
4	5		
		3	6

Rules for placing pairs in S_2, S_3 and S_4

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1	8		
		2	7
4	5		
		3	6

Rules for placing pairs in S_2, S_3 and S_4

R.7 Having placed elements in S_1 and S_2, 9 is placed in the only horse move position that is available from 1.

1	8		
		2	7
4	5		9
		3	6

Rules for placing pairs in S_2, S_3 and S_4

R.7 Having placed elements in S_1 and S_2, 9 is placed in the only horse move position that is available from 1.

Equivalent pairs horse moves			
H_1	H_2	H_3	H_4
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1	8		
		2	7
4	5		9
	10	3	6

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1	8		
	11	2	7
4	5		9
	10	3	6

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1	8		12
	11	2	7
4	5		9
	10	3	6

Rules for placing pairs in S_2, S_3 and S_4

R.7 Having placed elements in S_1 and S2, 9 is placed in the only horse move position that is available from 1.

R.8 Having placed elements in S_1, S2 and S_3, 13 is placed in the only horse move position that is available from 9.

Equivalent pairs horse moves			
H_1	**H_2**	**H_3**	**H_4**
(1,2)	(5,6)	(9,10)	(13,14)
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1	8	13	12
	11	2	7
4	5		9
	10	3	6

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1	8	13	12
14	11	2	7
4	5		9
	10	3	6

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4	5		9
15	10	3	6

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(3,4)	(7,8)	(11,12)	(15,16)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Analysis

Based on the aforesaid rules, it can be observed that:

- Having fixed a position for 1, there are four possible moves for placing 2. Thus the pair (1,2) can have four different configurations.
- For a fixed position of 1, there are three possible moves for placing 3.
- It becomes evident that there will remain only one position for placing 4 in accordance to the rules R4.

Thus at this stage we see that there are totally $4 \times 3 = 12$ possible configurations.

- For a given arrangement of 1,2,3 and 4, only two possible positions are available for 5 satisfying the aforesaid rule R.5. Thus at this stage we see that there are totally $12 \times 2 = 24$ possible configurations.
- Once the numbers 1, 2, 3, 4, and 5 are fixed, the rest of the square has only a unique solution for being a pan diagonal magic square.

Analysis

1			
		2	
	3		5
			4

1			
		2	
	3		
		5	4

1			
		2	
			3
	4	5	

1			
		2	
	5		3
	4		

1			
		2	
4	5		
		3	

1			
		2	
4			5
		3	

1			
			4
	3		5
		2	

1			
		5	4
	3		
		2	

1			
	4	5	
			3
		2	

1			
	4		
	5		3
		2	

1			
		3	
4	5		
		2	

1			
		3	
4			5
		2	

1			
		3	
	2		5
			4

1			
		3	
	2		
		5	4

1		4	
		5	
	2		3

1		4	
	2		3
		5	

1			
		5	4
	2		
		3	

1			
			4
	2		5
		3	

1			
		3	
	5		2
	4		

1			
		3	
			2
	4	5	

1			
	4	5	
			2
		3	

1			
	4		
	5		2
		3	

1		4	
		5	
	3		2

1		4	
	3		2
		5	

Properties of Nārāyaṇa Paṇḍita's pandiagonal magic squares of order 4

A study of the pandiagonal magic squares of order four, reveals that there are a few interesting properties observed in them.

- P1: The sum of elements in any 2 x 2 square across the torus of a pandiagonal magic square of order four yields the magic sum.
- P2: The sum of elements in any two cells that are separated by one cell in between them (*koṣṭhāntara*) along the diagonal on a pandiagonal magic square of order four yields the half magic sum.
- P3: For any given number of the arithmetic series that is chosen to fill the cells, the set of four numbers in the adjacent cells (*koṣṭhaikya*) along the row and column remain the same.

These properties are very useful to complete incomplete pandiagonal magic squares of order four as well as build them, in addition to the aforementioned algorithm.

Proof for P1

In any magic square the sum of the elements in rows and columns add up to the magic sum. Considering the sum of the first two rows and last two columns we have,

$$m11 + m12 + m13 + m14 + m21 + m22 + m23 + m24 = 2S \quad (1)$$

$$m13 + m23 + m33 + m43 + m14 + m24 + m34 + m44 = 2S \quad (2)$$

Equating (1) and (2) we get,

$$m11 + m12 + m21 + m22 = m33 + m34 + m43 + m44 \quad (3)$$

m11	m12	m13	m14
m21	m22	m23	m24
m31	m32	m33	m34
m41	m42	m43	m44

In a magic square, since the cyclic permutation of the rows and columns does not affect the magic sum S, (3) implies that the sum of the elements of any two 2×2 squares are equal. The leading and broken diagonal elements must independently add up to the magic sum. Thus we have

$$m11 + m22 + m33 + m44 = S \quad (4)$$

$$m21 + m12 + m43 + m34 = S \quad (5)$$

Adding (4) and (5) we get

$$(m11 + m12 + m21 + m22) + (m33 + m34 + m43 + m44) = 2S \quad (6)$$

From (1) and (3) it is evident that

$$m11 + m12 + m21 + m22 = S = m33 + m34 + m43 + m44 \quad (7)$$

Proof for P2

From P1 we know that

$$m11 + m12 + m21 + m22 = S \tag{8}$$

$$m21 + m22 + m31 + m32 = S \tag{9}$$

m11	m12	m13	m14
m21	m22	m23	m24
m31	m32	m33	m34
m41	m42	m43	m44

Equating (8) and (9) we get,

$$m11 + m12 = m31 + m32 \tag{10}$$

Now consider the sum of elements in the third row of the square M. We have,

$$m31 + m32 + m33 + m34 = S \tag{11}$$

Using (10) and (11) we get,

$$m11 + m12 + m33 + m34 = S \tag{12}$$

By definition, any broken diagonal of M must add up to the magic sum S. So we have,

$$m21 + m12 + m43 + m34 = S \tag{13}$$

Equating (12) and (13) we get,

$$m11 + m33 = m21 + m43 \tag{14}$$

Proof for P2

We have,

$$m11 + m33 = m21 + m43$$

Considering the sum of elements of the third column in M, we have:

$$m13 + m23 + m33 + m43 = S \tag{15}$$

Akin to (10), it can be easily shown that

$$m11 + m21 = m31 + m32 \tag{16}$$

m11	m12	m13	m14
m21	m22	m23	m24
m31	m32	m33	m34
m41	m42	m43	m44

Using (16) in (15) we get,

$$m11 + m21 + m33 + m43 = S \tag{17}$$

From (14) and (17) it is evident that

$$m11 + m33 = S/2 = m21 + m43 = S/2 \tag{18}$$

The property P2 is thus proved.

Set of numbers that get placed by horse moves

It has been stated by Vijayaraghavan (1941) that in a pandiagonal magic square of order 4, for any given number from the series of sixteen elements, the set of four other numbers that get placed by **the four horse move positions are always fixed** in all the 384 configurations. These four numbers, of course, can be arranged in $4! = 24$ ways.

These four numbers that get placed by the only four horse move positions from any number are:

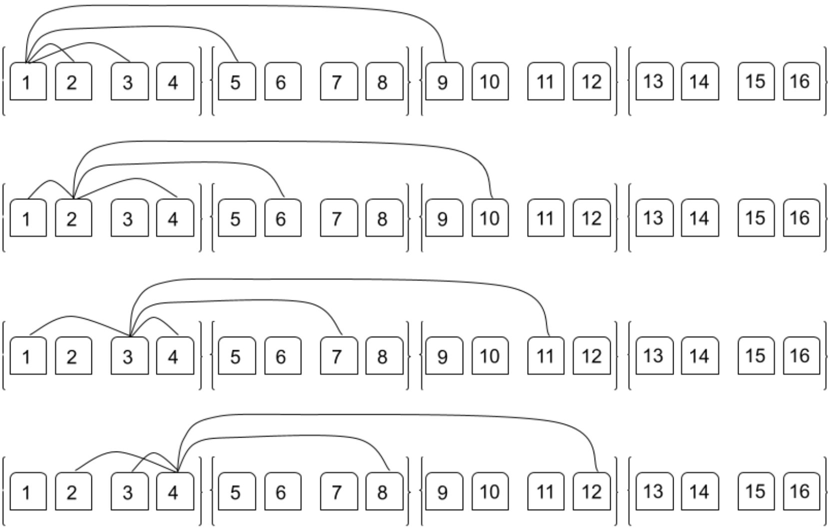
i the number that is in ***koṣṭhaikya*** within the pair (yugala),

ii the number that is in ***koṣṭhāntara*** within the pair of pairs (yamalāṅkayugalam),

iii the number that lies in the equivalent position with the **adjacent pair of pairs** (*yamalāṅkayugalam*) and

iv the number that lies in the equivalent position in the **conjugate pair of pairs** (*yamalāṅkayugalam*).

Set of numbers that get placed by horse moves



Set of numbers that get placed by horse moves

Given number	Set of numbers positioned by horse moves			
	Within the pair	Within pair of pairs	With adjacent pair of pairs	With the next to adjacent pair of pairs
1	2	3	5	9
2	1	4	6	10
3	4	1	7	11
4	3	2	8	12
5	6	7	1	13
6	5	8	2	14
7	8	5	3	15
8	7	6	4	16
9	10	11	13	1
10	9	12	14	2
11	12	9	15	3
12	11	10	16	4
13	14	15	5	9
14	13	16	6	10
15	16	13	7	11
16	15	14	8	12

All squares with a fixed position for the first element

S1

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

S2

1	14	7	12
8	11	2	13
10	5	16	3
15	4	9	6

S3

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

S4

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

S5

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

S6

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

S7

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

S8

1	12	7	14
15	6	9	4
10	3	16	5
8	13	2	11

S9

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

S10

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

S11

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

S12

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

S13

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

S14

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

S15

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

S16

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

S17

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

S18

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

S19

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

S20

1	12	6	15
8	13	3	10
11	2	16	5
14	7	9	4

S21

1	8	10	15
14	11	5	4
7	2	16	9
12	13	3	6

S22

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

S23

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

S24

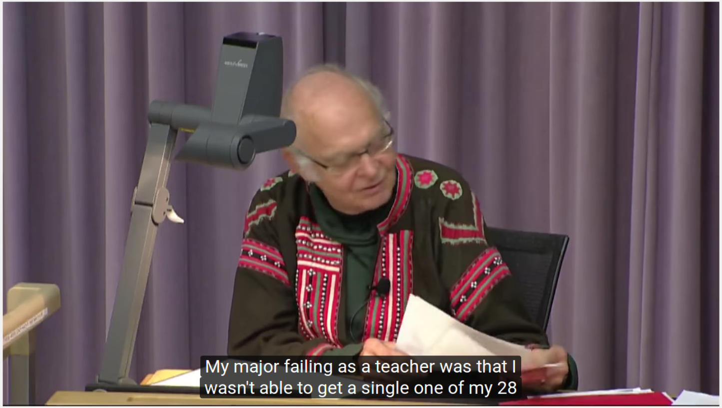
1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

Concluding remarks

- In this study, we have set the context and highlighted the motivation for studying the *turagagati* method of constructing 4 x 4 pandiagonal magic squares as elucidated by Nārāyaṇa Paṇḍita.
- The verses that present this method in *Gaṇitakaumudī* allow scope for more than one interpretation.
- The algorithm can be derived either by placing the pairs from a sequence only through horse moves or by considering moves in addition to horse moves.
- Of foremost importance to Indian mathematicians is simplicity and optimised set of rules. It is clear that the algorithm that employs only the horse moves by taking pairs in order and by jumping the order, within the pair, within pair of pairs and across pair of pairs, seems to be the more elegant.
- A very interesting facet of this algorithm is that the pandiagonal squares can be generated by choosing a single arithmetic sequence with sixteen elements or with four arithmetic sequences consisting a pair of pairs, with the constraint that the first elements of the pair of pairs satisfy – $a_1 + d_1 = b_1 + c_1$.

The thrill in investigating historical material

An interesting observation by Donald Knuth – the author of *The Art of Computing*



My major failing as a teacher was that I wasn't able to get a single one of my 28 PhD students to realize what a thrill it is to work on source material!

Concluding Remarks

- In 1969 Sir C V Raman observes:

> We have, I think, developed an inferiority complex. I think what is needed in India today is the destruction of that defeatist spirit. We need a spirit of victory, a spirit that will carry us to our rightful place under the sun; a spirit which will recognise that we, as inheritors of a proud civilisation, are entitled to our rightful place on the planet. If that indomitable spirit were to arise,[a] nothing can hold us back from achieving our rightful destiny.
>
> [a]The Upanisadic passage – उत्तिष्ठत जाग्रत प्राप्य वरान्निबोधत – essentially does this.

- Raman brings out an extremely important point here. The inferiority complex, residing in us as a parasite, without our knowledge, has been inhibiting us for centuries!
- One way (if not the only way) to get rid of this problem is to make the citizens of our nation to shed way the cultivated ignorance by making them aware of their own scientific heritage.
- Then as Raman says, nothing can hold us back!

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Thanks!

Thank You !